COMPATIBILITY AND SUMMABILITY OF OBSERVABLES IN FUZZY QUANTUM SPACES

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ABSTRACT. Using ideas of fuzzy set theory, we study the property of compatibility and the possibility of defining a sum of observables in fuzzy quantum spaces.

1. Introduction

In the last period, it appears an axiomatic model, fuzzy quantum spaces [8, 10, 2], which may serve, for example, as a new basis for description of quantum mechanical events. It connects two different areas of mathematics: quantum logic theory, and fuzzy set theory. For quantum logic theory [14], an important example is a q- σ -*algebra*, suggested by Suppes [12], that is a non-empty collection $\mathcal{Q}$ of subsets of a non-void set X which is closed with respect to the complementation and to the countable unions of disjoint subsets. On the other hand, for fuzzy set theory, Piasecki [6] suggested a model of a soft- σ -algebra, which is a non-empty system M of fuzzy sets of a set $X \neq \emptyset$, that is, M is a system of functions defined on X with values in the interval $[0, 1]$ such that

(1.1) M contains the constant function $1(x) = 1$, $x \in X$;
(1.2) if $f \in M$, then $f^{\perp} := 1 - f \in M$;
(1.3) if $1/2(x) := 1/2$, $x \in X$, then $1/2 \notin M$;
(1.3)* if $f_n \in M$, $n \geq 1$, then $\bigcup_{n=1}^{\infty} f_n := \sup_n f_n \in M$.

Due to [6], two fuzzy sets f and g are called *weakly separated* if $f \leq 1 - g$; and we see that in both above models there are many remarkable similarities. Therefore, Riečan [8] and one of the present authors [10], using (1.1)–(1.3)*, proposed to study a model of F-quantum spaces, and Pykacz [7] suggested to assume that M is closed only with respect to union of separated fuzzy sets. Therefore, by a *fuzzy quantum space* we shall understand any couple (X, M), where $M \subset [0, 1]^X$, which fulfills the conditions (1.1)–(1.3) and

$$\text{if } f_n \leq 1 - f_m,\ n \neq m, \text{then } \bigcup_n f_n \in M, \tag{1.4}$$

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and the set M is said to be a *fuzzy-q-σ-algebra.*

In particular, if Q is a q-σ-algebra, and $M = \{I_A : A \in Q\}$, where I_A is the indicator of a subset A, then (X, M) is a fuzzy quantum space.

2. Fuzzy Quantum Spaces

In the present, fuzzy quantum spaces are intensively studied by many authors [4, 5, 8, 10, 13]. They mainly investigate problems which are important namely from quantum logic point of view. Therefore, there are introduced such notions as states, observables, compatibility, summability of observables, mean values, etc.

Due to the terminology of quantum logic theory, we say that two fuzzy events (= fuzzy sets) f and g from M are *orthogonal*, and we write $f \perp g$, if they are weakly separated. By an *orthogonal complement* of a fuzzy set a we mean the fuzzy set $a^\perp := 1 - a$. The complementation "$\perp$" connects the fuzzy union $\cup$ and the fuzzy intersection $\cap$, which is defined as $\bigcap_i a_i := \inf_i a_i$, by the following manner

$$(\bigcup_n a_n)^\perp = \bigcap_n a_n^\perp,$$
$$(\bigcap_n a_n)^\perp = \bigcup_n a_n^\perp,$$

which means that if one side exists in M, so exists the second one, and both are equal.

Let (X, M) be a fuzzy quantum space. A non-empty set $A \subseteq M$ is said to be a *Boolean σ-algebra* of the fuzzy quantum space (X, M) if

(2.1) There are minimal and maximal elements 0_A and 1_A in A such that, for any $f \in A$, $0_A \leq f \leq 1_A$ and $f \cup f^\perp = 1_A$.

(2.2) With respect to $\cup, \cap, \perp, 0_A$, and 1_A, A is a Boolean σ-algebra (in the sense of Sikorski [11]).

It is clear that $0_A \neq 1_A$.

If a is a fuzzy set belonging to M, then $M_a = \{a \cap a^\perp, a, a^\perp, a \cup a^\perp\}$ is a Boolean σ-algebra of (X, M) with the minimal and maximal elements $a \cap a^\perp$ and $a \cup a^\perp$, respectively.

If A and B are two Boolean σ-algebras of (X, M) with a non-empty intersection, then $1_A = 1_B$ and $0_A = 0_B$. Indeed, let $a \in A \cap B$, then $1_A = a \cup a^\perp = 1_B$.

Let $\mathcal{B}(\mathbb{R})$ be the Borel σ-algebra of subsets of the real line $\mathbb{R}$. By an *observable* of (X,M) we mean a mapping $x:\ \mathcal{B}(\mathbb{R}) \to M$ such that

(2.3) $x(A^c) = x(A)^\perp,\ A \in \mathcal{B}(\mathbb{R}),\ (A^c := \mathbb{R} \setminus A)$;

(2.4) $x(A) \perp x(B)$ if $A \cap B = \emptyset,\ A, B \in \mathcal{B}(\mathbb{R})$;

(2.5) $x(\bigcup_i A_i) = \bigcup_i x(A_i)$, if $A_i \cap A_j = \emptyset$ for $i \neq j,\ A_i \in \mathcal{B}(\mathbb{R}),\ i \geq 1$.

If a is a fuzzy set from M, then the mapping x_a defined via

$$x_a(E) = \begin{cases} a \cap a^\perp & \text{if } 0,1 \notin E, \\ a^\perp & \text{if } 0 \in E,\ 1 \notin E, \\ a & \text{if } 0 \notin E,\ 1 \in E, \\ a \cup a^\perp & \text{if } 0,1 \in E, \end{cases} \tag{2.6}$$

for any $E \in \mathcal{B}(\mathbb{R})$, is an observable of (X,M), and it plays the role of the indicator of the fuzzy set $a \in M$.

It is simple to verify that if x is an observable of (X,M) and $f:\mathbb{R} \to \mathbb{R}$ is a Borel measurable, then $f(x) := x \circ f^{-1}$ is an observable of (X,M), too.

Denote by $\mathcal{R}(x)$ the range of an observable x, that is $\mathcal{R}(x) = \{x(E):\ E \in \mathcal{B}(\mathbb{R})\}$. Then $\mathcal{R}(x)$ is a Boolean σ-algebra of (X,M) with the minimal and maximal elements $x(\emptyset)$ and $x(\mathbb{R})$, respectively, and $x(\bigcup_i E_i) = \bigcup_i x(E_i)$ for any sequence $\{E_i\}$ of Borel sets.

Let $\mathcal{Q}$ be a q-σ-algebra of subsets of a set X and let $\phi:\ X \to \mathbb{R}$ be a $\mathcal{Q}$-measurable mapping, that is $\phi^{-1}(E) \in \mathcal{Q}$ for every $E \in \mathcal{B}(\mathbb{R})$. The transformation $x:\ \mathcal{B}(\mathbb{R}) \to M = \{I_A:\ A \in Q\}$ defined via

$$x(E) = I_{\phi^{-1}(E)}, \quad E \in \mathcal{B}(\mathbb{R}), \tag{2.7}$$

is an observable of the fuzzy quantum space (X,M). Conversely, for every observable x of (X,M) there is a unique real-valued mapping $\phi:\ X \to \mathbb{R}$ such that (2.7) holds.

A *state* of (X,M) is a mapping $m: M \to [0,\ 1]$ such that

(2.8) $m(f \cup f^\perp) = 1$ for every $f \in M$;

(2.9) $m(\bigcup_i f_i) = \sum_i m(f_i)$ for any sequence $\{f_i\}$ of mutually orthogonal fuzzy sets from M.

It is simple to justify that if x is an observable of (X,M) and m is a state, that $m_x :\mapsto m(x(E)),\ E \in \mathcal{B}(\mathbb{R})$, is a probability measure on $\mathcal{B}(\mathbb{R})$, and we define a *mean value* of x in the state m as the expression $m(x) = \int x\,\mathrm{d}m$ defined via

$$m(x) = \int_{\mathbb{R}} t\,\mathrm{d}m_x(t) \tag{2.9}$$

(if the integral on the right-hand side exists and is finite).

3. Compatibility

In accordance with the theory of quantum logics, we say that two elements $f, g \in M$ are (i) *compatible*, and we write $f \leftrightarrow g$, if $f \cap g$, $f \cap g^\perp$, $f^\perp \cap g \in M$, and $f = f \cap g \cup f \cap g^\perp$, $g = f \cap g \cup f^\perp \cap g$; and (ii) *strongly compatible*, and write $f \overset{s}{\leftrightarrow} g$, if $f \leftrightarrow g \leftrightarrow f^\perp \leftrightarrow g^\perp \leftrightarrow f$. It is evident that if $f \leftrightarrow g$, then $f \cup g \in M$, and $f \overset{s}{\leftrightarrow} f^\perp$, $f \cap f^\perp \overset{s}{\leftrightarrow} f \overset{s}{\leftrightarrow} f \cup f^\perp \overset{s}{\leftrightarrow} f \cap f^\perp$. We note that if $f \leftrightarrow g$, then it does not imply $f \overset{s}{\leftrightarrow} g$, in general [2].

We say that two non-empty sets A and B of M are *compatible* (*strongly compatible*) if $a \leftrightarrow b$ ($a \overset{s}{\leftrightarrow} b$) for all $a \in A, b \in B$. For Boolean algebras A and B of (X, M) both kinds of compatibility coincide.

A non-void subset A of M is said to be f-*compatible* if, for all $f_1, \dots, f_{n+1} \in A$, we have (i) $b_1 := f_1 \cap \dots \cap f_n \cap f_{n+1} \in M$; $b_2 := f_1 \cap \dots \cap f_n \cap f_{n+1}^\perp \in M$; (ii) $b_1 \cap b_2 = f_1 \cap \dots \cap f_n$. The subset A of M is *strongly* f-*compatible* if $A \cup A^\perp$ is f-compatible, where $A^\perp = \{f^\perp : f \in A\}$.

The following principal result for compatibility has been proved in [2]:

THEOREM 1. *Let A be a non-empty set of a fuzzy quantum space (X, M). The following statements are equivalent:*

(i) *A is strongly f-compatible.*

(ii) *There is a Boolean σ-algebra B of (X, M) containing A.*

In [2, 3, 10], it has been proved that if M is closed with respect to union of all sequences of fuzzy sets, then $A \subseteq M$ is strongly f-compatible iff

$$a \cup a^\perp = b \cup b^\perp, \quad a, b \in A. \tag{3.0}$$

The following criteria will be useful for us:

LEMMA 1. (i) *Let $a_1, \dots, a_n$ be mutually orthogonal elements of M. Then $\{a_1, \dots, a_n\}$ is strongly f-compatible iff*

$$a_1 \cup a_1^\perp = a_2 \cup a_2^\perp = \dots = a_n \cup a_n^\perp. \tag{3.1}$$

(ii) *Let $b_1 \le \dots \le b_n$. Then $\{b_1, \dots, b_n\}$ is strongly f-compatible iff*

$$b_1 \cup b_1^\perp = b_2 \cup b_2^\perp = \dots = b_n \cup b_n^\perp. \tag{3.1}$$

P r o o f . It is evident that the strong f-compatibility entails (3.1) and (3.2), respectively.

(i) Suppose (3.1), and put $0_a = a_1 \cap a_1^\perp$ and $1_a = a_1 \cup a_1^\perp$. Then $0_a = 0_a \cap \cdots \cap 0_a = (a_1 \cap a_1^\perp) \cap \cdots \cap (a_n \cap a_n^\perp) = a_1 \cap \cdots \cap a_n \in M$. For any $b \in M$ we define $b^0 = b^\perp$ and $b^1 = b$. Therefore, for any $(i_1, \ldots, i_n) \in \{0,1\}^n$ we have

$$a_1^{i_1} \cap \cdots \cap a_n^{i_n} = \begin{cases} (a_1 \cup \cdots \cup a_n)^\perp & \text{if } i_j = 1,\ j = 1, \ldots n, \\ a_k & \text{if } i_k = 1 \text{ and } i_j = 0 \text{ for } j \neq k, \\ 0_a & \text{if at least two } i_j, i_k = 1. \end{cases}$$

Therefore, $a_1^{i_1} \cap \cdots \cap a_{n-1}^{i_{n-1}} \cap a_n^{i_n} \cup a_1^{i_1} \cap \cdots \cap a_{n-1}^{i_{n-1}} \cap (a_n^{i_n})^\perp = a_1^{i_1} \cap \cdots \cap a_{n-1}^{i_{n-1}}$.

(ii) Define $a_1 = b_1$, $a_k = b_k \cap b_{k-1}^\perp$, $k = 2, \ldots, n$, and put $0_b = b_1 \cap b_1^\perp$, $1_b = b_1 \cup b_1^\perp$. Then $a_1, \ldots, a_n$ are mutually orthogonal, and for any $i = 2, \ldots, n$ $a_i \cup a_i^\perp = b_i \cap b_{i-1}^\perp \cup b_{i-1} \cup b_i^\perp = 0_b \cup b_{i-1} \cup b_i^\perp = b_{i-1} \cup b_i^\perp \leq b_i \cup b_i^\perp = 1_b$, and simultaneously $b_{i-1} \cup b_i^\perp \geq b_{i-1} \cup b_{i-1}^\perp = 1_b$, so that (3.1) holds. Therefore, $\{a_1, \ldots, a_n\}$ is strongly f-compatible.

Calculate $b_2 = b_2 \cap 1_b = b_2 \cap (b_1 \cup b_1^\perp) = b_1 \cup b_2 \cap b_1^\perp = a_1 \cup a_2$. Using the mathematical induction, we have $b_k = b_k \cap 1_b = b_k \cap (b_{k-1} \cup b_{k-1}^\perp) = b_{k-1} \cup b_k \cap b_{k-1}^\perp = a_1 \cup \cdots \cup a_{k-1} \cup a_k$. Therefore, $b_1, \ldots, b_k \in B$, and $\{b_1, \ldots, b_n\}$ is strongly f-compatible.

The same result may be also proved more elementary: It is simple to see that for any $(i_1, \ldots, i_n) \in \{0,1\}^n$ we have

$$b_1^{i_1} \cap \cdots \cap b_n^{i_n} = \begin{cases} b_1 & \text{if } \ i_k = 1,\ k = 1, \ldots, n, \\ 0_b & \text{if there are } i_k = 1 \text{ and } i_t = 0, \\ b_n^\perp & \text{if } \ i_k = 0,\ k = 1, \ldots, n, \end{cases}$$

which entails the strong f-compatibility. ■

The simple corollary of the last criterion is the following:

EXAMPLE 1. *Let $\{a_i\}_{i=1}^\infty$ be a sequence of mutually orthogonal strongly f-compatible elements from M. Put $a_o = (\bigcup_{i=1}^\infty a_i)^\perp$. Then a mapping x defined via*

$$x(E) = \begin{cases} a_o \cap a_o^\perp & \textit{if } \ 0, 1, \cdots \notin E, \\ \bigcup\{a_i :\ i \in E\} & \textit{otherwise}, \end{cases} \qquad E \in \mathcal{B}(\mathbb{R}),$$

is an observable of (X, M).

Now we present a characterization of observables of (X, M).

THEOREM 2. *Let x be an observable of a fuzzy quantum space (X, M), and let*

$$B_x(t) = x((-\infty, t)), \ t \in \mathbb{R}. \tag{3.3}$$

Then the system $\{B_x(t) : \ t \in \mathbb{R}\}$ fulfills the following conditions:

(3.4) $B_x(s) \leq B_x(t)$ *if* $s \leq t$;

(3.5) $\bigcup_t B_x(t) = a$;

(3.6) $\bigcap_t B_x(t) = a^\perp$;

(3.7) $\bigcup_{t<s} B_x(t) = B_x(s)$ *for any* $s \in \mathbb{R}$;

(3.8) $B_x(t) \cup B_x(t)^\perp = a$ *for any* $t \in \mathbb{R}$,

where $a = x(\mathbb{R})$. Conversely, if a system $\{B(t) : \ t \in \mathbb{R}\}$ of fuzzy sets from M fulfills the conditions (3.4)–(3.8) *for some $a \in M$, then there is a unique observable x such that $B_x(t) = B(t)$ for any $t \in \mathbb{R}$ and $x(\mathbb{R}) = a$.*

P r o o f . The necessity is evident. For sufficiency it is easy to see that according to (3.4), (3.8) and (ii) of Lemma 1, $\{B(t) : \ t \in \mathbb{R}\}$ is a system of strongly f-compatible elements. In view of Theorem 1, there exists a Boolean σ-algebra B of (X, M) containing all $B(t)$'s, and now the proof follows the same ideas as that of Theorem 2.3 in [5]. ■

We say that a system of observables, $\{x_t : \ t \in T\}$, is *compatible, strongly compatible,* f-*compatible, strongly* f-*compatible* if $\bigcup_{t \in T} \mathcal{R}(x_t)$ is such.

THEOREM 3. *A sequence $\{x_n\}$ of observables of (X, M) is f-compatible if and only if there is a sequence of Borel measurable, real-valued functions $\{f_n\}$ and an observable x such that*

$$x_n = f_n(x), \ n \geq 1. \tag{3.9}$$

P r o o f . The necessity implies easily the f-compatibility of $\{x_n\}$. Conversely, let $\{x_n\}$ be any sequence of f-compatible observables. In view of Theorem 1, there is a Boolean σ-algebra A of (X, M) containing all ranges of x_n, $n \geq 1$. Due to the result of Varadarajan [14], we obtain (3.9) ■

This theorem enables us to built up a so-called calculus for f-compatible observables. Therefore, we may define, for example, the sum $x_1 \dot{+} \cdots \dot{+} x_n$ of f-compatible observables via

$$x_1 \dot{+} \cdots \dot{+} x_n = (f_1 + \cdots + f_n)(x), \tag{3.10}$$

or multiplication $x_1 \cdot x_2 = (f_1 \cdot f_2)(x)$, etc.

We recall that these expressions do not depend on the choice of x and $f_1, \ldots, f_n$.

4. Summability of Observables

In classical probability theory, it is simple to define the sum of two random variables. Since for us observables are analogues of random variables, we try to present the definition of sum of two observables.

We say that two observables x and y are summable with a sum $z = x + y$ if, for any $t \in \mathbb{R}$, the element

$$B_{x+y}(t) = \bigcup_{r\in\mathbb{Q}} (B_x(r) \cap B_y(t-r)), \tag{4.1}$$

where $\mathbb{Q}$ is the set of all rational numbers in $\mathbb{R}$, exists in M. In order to be this definition correct, it is necessary to show that a system $\{B_{x+y}(t) : t \in \mathbb{R}\}$ fulfills the conditions (3.4)–(3.8) of Theorem 2.

THEOREM 4. *If, for two observables x and y, the elements $B_{x+y}(t)$ exist in M for all $t \in \mathbb{R}$, then $\{B_{x+y}(t) : t \in \mathbb{R}\}$ determines a unique observable z of (X, M), moreover, $z(\mathbb{R}) = x(\mathbb{R}) \cap y(\mathbb{R})$, and $x + y = y + x$.*

Proof. It is same as that of Theorem 2.3 in [5], and it uses the following lemma:

LEMMA 2. *Let $\mathbb{S}$ be a countable dense set in $\mathbb{R}$ and let x and y be summable observables of (X, M). Define*

$$B^{\mathbb{S}}_{x+y}(t) = \bigcup_{s\in\mathbb{S}} (B_x(s) \cap B_y(t-s)), \tag{4.2}$$

then $B^{\mathbb{S}}_{x+y}(t) \in M$ for any $t \in \mathbb{R}$, and

$$B^{\mathbb{S}}_{x+y}(t) = B_{x+y}(t) \text{ for any } t \in \mathbb{R}. \tag{4.3}$$

Proof. We may show that if $t_n \nearrow t$, $t_n \in \mathbb{S}$, then $B^{\mathbb{S}}_{x+y}(t) = \bigcup_n B^{\mathbb{S}}_{x+y}(t_n)$. Indeed,

$$\bigcup_n B^{\mathbb{S}}_{x+y}(t_n) = \bigcup_n \bigcup_{s\in\mathbb{S}} (B_x(s) \cap B_y(t_n - s)) =$$

$$\bigcup_{s\in\mathbb{S}} (B_x(s) \cap \bigcup_n B_y(t_n - s)) = \bigcup_{s\in\mathbb{S}} (B_x(s) \cap B_y(t-s)).$$

Now let n be any integer, then, for each $s \in \mathbb{S}$, there is $r = r(s) \in \mathbb{Q}$ such that $s < r < s + 1/n$. Therefore, $B_x(s) \cap B_y(t - 1/n - s) \leq B_x(r) \cap B_y(t - r)$ and $B^{\mathbb{S}}_{x+y}(t - 1/n) \leq B_{x+y}(t)$, $B^{\mathbb{S}}_{x+y}(t) = \bigcup_n B^{\mathbb{S}}_{x+y}(t - 1/n) \leq B^{\mathbb{S}}_{x+y}(t)$,

Similarly we show that $B_{x+y}(t) \leq B^{\mathbb{S}}_{x+y}(t)$. ■

In [5], it has been proved that if M is closed with respect to the union of any sequence of fuzzy sets, then any two observables of (X, M) are summable. In below, we show that, for general fuzzy quantum spaces, this is not true, in general.

EXAMPLE 2. *Let $X = \{1,2,3,4\}$ and let M consist of the characteristic functions of all subsets of X with an even number of elements. Let $\xi = I_{\{1,2\}}$, $\eta = I_{\{1,3\}}$. Define two observables x and y via $x(E) = I_{\xi^{-1}(E)}$, $y(F) = I_{\eta^{-1}(F)}$, $E, F \in \mathcal{B}(\mathbb{R})$. Then x and y are not summable observables of (X, M).*

PROPOSITION 3. *If x and y are f-compatible observables of (X, M), then they are summable, and*

$$x \dot{+} y = x + y. \tag{4.4}$$

Proof. Due to Theorem 3, there exist an observable u of (X, M) and two Borel real-valued functions f and g such that $x = f(u)$, $y = g(u)$. Calculate

$$\begin{aligned} B_{x+y}(t) &= \bigcup_{r\in\mathbb{Q}} (B_x(r) \cap B_y(t-r)) = \bigcup_{r\in\mathbb{Q}} u(f^{-1}(-\infty, r)) \cap u(g^{-1}(-\infty, t-r)) \\ &= u(\bigcup_{r\in\mathbb{Q}} (f^{-1}(-\infty, r) \cap g^{-1}(-\infty, t-r))) = u((f+g)^{-1}(-\infty, t)) \\ &= B_{x\dot{+}y}(t) \in M. \ ■ \end{aligned}$$

Let x be an observable of (X, M). The *spectrum* of x is a set $\sigma(x)$ defined via

$$\sigma(x) = \bigcap \{C : \ C \text{ is closed and } x(C) = x(\mathbb{R})\}. \tag{4.5}$$

Then $\sigma(x) \in \mathcal{B}(\mathbb{R})$ and $x(\sigma(x)) = x(\mathbb{R})$. An observable x is *simple* if $\sigma(x) = \{t_1, \ldots, t_n\}$.

THEOREM 5. *Suppose that either x and y are compatible or $(1.3)^*$ holds in M. Let m be a state on (X, M). Then the following mean value additivity holds*

$$m(x + y) = m(x) + m(y) \tag{4.6}$$

whenever $m(x)$ and $m(y)$ exist and are finite.

Proof. For the first case, (4.6) holds due to (4.4) and the following considerations

$$\begin{aligned}
&m(x+y) = m(x\dot{+}y) = \int_{\mathbb{R}} t\,\mathrm{d}m_{x\dot{+}y}(t) = \int_{\mathbb{R}} t\,\mathrm{d}m_{(f+g)(u)}(t) = \\
&\int_{\mathbb{R}} (f(t)+g(t))\,\mathrm{d}m_u(t) = \int_{\mathbb{R}} f(t)\,\mathrm{d}m_u(t) + \int_{\mathbb{R}} g(t)\,\mathrm{d}m_u(t) = \\
&\int_{\mathbb{R}} t\,\mathrm{d}m_x(t) + \int_{\mathbb{R}} t\,\mathrm{d}m_y(t) = m(x)+m(y).
\end{aligned}$$

For the second case we define two new observables $\overline{x}$ and $\overline{y}$ via

$$\begin{aligned}
\overline{x}(E) &= x(E)\cap y(\mathbb{R}) \cup x(\emptyset)\cup y(\emptyset),\\
\overline{y}(E) &= y(E)\cap x(\mathbb{R}) \cup x(\emptyset)\cup y(\emptyset),
\end{aligned}$$

for any $E \in \mathcal{B}(\mathbb{R})$. Then $\overline{x}(\mathbb{R}) = x(\mathbb{R})\cap y(\mathbb{R}) = \overline{y}(\mathbb{R})$. Put $1_h = x(\mathbb{R})\cap y(\mathbb{R})$ and $0_h = 1_h^{\perp}$. Calculate

$$\begin{aligned}
&B_{\overline{x}+\overline{y}}(t) = \bigcup_{r\in\mathbb{Q}} (B_{\overline{x}}(r)\cap B_{\overline{y}}(t-r)) = \\
&\bigcup_{r\in\mathbb{Q}} (B_x(r)\cap 1_h \cup 0_h)\cap((B_y(t-r)\cap 1_h\cup 0_h)) = \\
&\bigcup_{r\in\mathbb{Q}} (B_x(r)\cap B_y(t-r)\cap 1_h \cup 0_h\cap B_y(t-r)\cap 1_h \cup B_x(r)\cap 1_h\cap 0_h\cup 0_h) = \\
&\bigcup_{r\in\mathbb{Q}} (B_x(r)\cap B_y(t-r)) \cup \bigcup_{r\in\mathbb{Q}} 0_h\cap B_y(t-r)\cup \bigcup_{r\in\mathbb{Q}} 0_h\cap B_x(r)\cup 0_h = \\
&B_{x+y}(t)\cup 0_h\cap y(\mathbb{R})\cup 0_h\cap x(\mathbb{R})\cup 0_h) = B_{x+y}(t)\cup 0_h = B_{x+y}(t).
\end{aligned}$$

This yields that $m_x = m_{\overline{x}}$, $m_y = m_{\overline{y}}$ and $m_{x+y} = m_{\overline{x}+\overline{y}}$. Using the first part of the present assertion for compatible observables (see (3.0)), we have (4.6). ■

The second part of the present theorem has been also proved by different methods in [5] and [9].

5. An Open Problem

For q-σ-observables the additivity of mean values has been solved for many years. Dravecký and Šípoš [1] found an example of two summable observables (=random variables) for which the additivity of mean value does not hold:

Put $X = \mathbb{R}^2$, and, for any $n = 0, \pm 1, \pm 2, \cdots$, define $f_n = I_{(n,n+1]\times\mathbb{R}}$, $g_n = I_{\mathbb{R}\times(n,n+1]}$, $H_{ij} = (i, i+1] \times (j, j+1]$, $H_n = \bigcup\{H_{ij} : i + j = n\}$, $h_n = I_{H_n}$. Let M be a fuzzy-q-σ-algebra generated by all $f_n, g_n h_n$. Let m be a state such that $m(f_o) = m(g_o) = m(h_1) = 1$. Define observables z, y, z via $x(\{n\}) = f_n$, $y(\{n\}) = g_n$, $z(\{n\}) = h_n$, $n = 0, \pm 1, \pm 2, \ldots$. Then x and y are summable with the sum $z = x + y$, but $m(z) = 1$, and $m(x) = 0 = m(y)$.

We recall that their spectra were uncountable. For long time the additivity for summable simple variables has been solved. Definitely it has been proved by Zerbe and Gudder [16] (see also [15]).

PROBLEM. Inspiring above-said, we suggest to prove (4.6) for any pair of summable simple observables in any state for a fuzzy quantum space (X, M).

REFERENCES

[1] J. Dravecký, J. Šípoš: *On the additivity of Gudder integral*, Math. Slovaca **30** (1980), 299-303.

[2] A. Dvurečenskij, F. Chovanec: *Fuzzy quantum spaces and compatibility*, Inter. J. Theor. Phys. **27** (1988), 1069–1082.

[3] A. Dvurečenskij, B. Riečan: *On joint observables for F-quantum spaces*, Busefal **35** (1988), 10–14.

[4] A. Dvurečenskij, A. Tirpáková: *A note on a sum of observables in F-quantum spaces and its properties*, Busefal **35** (1988 132–137).

[5] A. Dvurečenskij, A. Tirpáková: *Sum of observables in fuzzy quantum spaces*, Aplikace matem. **37** (1992), 40–50.

[6] K. Piasecki: *Probability of fuzzy events defined as denumerable additivity measure*, Fuzzy Sets and Systems **17** (1985), 271–284.

[7] J. Pykacz: *Quantum logics and soft fuzzy probability spaces*, Busefal **32** (1987), 150–157.

[8] B. Riečan: *A new approach to some notions of statistical quantum mechanics*, Busefal **35** (1988), 4–6.

[9] B. Riečan: *On mean value in F-quantum spaces*, Aplikace matem. **35** (1989), 209–214.

[10] B. Riečan, A. Dvurečenskij: *Randomness and fuzziness*, Proceedings of the First Joint IFSA-EC and EURO-WG Workshop on Progress in Fuzzy Sets in Europe, Nov. 25-27, 1986, Warsaw, 1988, pp. 321–326.

[11] R. Sikorski: *Boolean Algebras*, Springer Verlag, Heidelberg, 1964.

[12] P. Suppes: *The probability argument for a non-classical logic of quantum mechanics*, Phil. Science **33** (1966), 14–21.

[13] A. Tirpáková: *On a sum of observables in F-quantum spaces and its application to convergence theorems*, Proceedings of the First Winter School on Measure Theory, Liptovský Ján, Jan. 10-15, 1988, (1988), pp. 161–166.

[14] V.S. Varadarajan: *Geometry of Quantum Theory*, Van Nostrand, New York, 1968.

[15] M. Navara: *When is the integration on quantum probability spaces additive ?*, Real Anal. Exchange **14** (1988/89), 228–234.

[16] J. Zerbe, S.P. Gudder: *Additivity of integrals on generalized measure spaces*, J. Comb. Theor. A **39** (1985), 42–51.

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