THE PSEUDO INVERTIBLE F-MATRIX

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ABSTRACT

The concept of the pseudo invertible F-matrix was introduced in this paper. And we proved that the A is a pseudo invertible F-matrix if and only if the A is an orthogonal F-matrix.

Keywords: Permutation matrix,Pseudo invertible F-matrix, Orthogonal F-matrix

In this paper we continue to use definitions and notations in reference [1].

As everyone knows, a permutation matrix P is a Boolean square matrix which contains exactly one elements of every row and every column of P, and exception is 0.

Theorem 1 P is a nth order permutation matrix if and only if

$$P'P = PP' = I_n.$$

Proof. $\Rightarrow$ If P is a nth order permutation matrix, show easy by computation that $P'P = PP' = I_n$.

$\Leftarrow$ If $PP' = P'P = I_n$. We use proof by contradiction. Suppose P is not a permutation matrix.

(a) If there is a row in P, and not only is contained one, but also is contained two ones,and conception is 0. Without loss of generality, we may let first row of P be

$$(1 \quad 1 \quad 0 \ \dots \ 0).$$

Since $PP' = I_n$, and first row in I_n is $(1 \quad 0 \ \dots \ 0)$,then we must let that

$$P' = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & * & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & * & \dots & * \end{pmatrix},$$

where "*" are several numbers in $\{0,1\}$. Thus

$$P'P=\begin{pmatrix}1 & 1 & 0 & \dots & 0\\ 1 & 1 & 0 & \dots & 0\\ 0 & 0 & * & \dots & *\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & * & \dots & *\end{pmatrix} \neq I_n,$$

this is a contradiction. Therefore, in every row of P has at best one.

(b) If there is a row in P which in the row of A does not contain 1, then $PP' \neq I_n$. this is a contradiction, still. Thus in every row of P has at least one.

To sum up, in every row of P is exactly contained one, and exception is 0.

Analogously we may prove that every column of P contains exactly one, and exception is 0.

Therefore P is a permutation matrix.

<u>Definition 2</u> Let $A = (a_{ij})_{n\times n}$ be a F-matrix. If there is a $\lambda \in (0,1]$ and a F-matrix $B = (b_{ij})_{n\times n}$, so that

(1) $$A_\lambda B_\lambda = B_\lambda A_\lambda = I_n,$$

then A is called a pseudo invertible F-matrix. And B is called a pseudo inverse matrix of A. And A_λ is called the permutation matrix of A.

<u>Example 1</u> For $A = \begin{pmatrix}.4 & .2 & .2\\ .3 & .5 & .3\\ .3 & .2 & .6\end{pmatrix}$ and $B = \begin{pmatrix}.4 & .1 & .2\\ .3 & .4 & .3\\ .2 & .2 & .6\end{pmatrix}$,

we fetch $\lambda = 0.4$, then

$$A_{0.4} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \text{ and } B_{0.4} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Thus $A_{0.4}B_{0.4} = B_{0.4}A_{0.4} = I_3$. Therefore A is a pseudo invertible F-matrix.

<u>Example 2</u> Let $$C = \begin{pmatrix}.4 & .4 & .2\\ .3 & .4 & .3\\ .3 & .2 & .4\end{pmatrix},$$

by

$$C_{0.2} = \begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}, \qquad C_{0.3} = \begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 1\\ 1 & 0 & 1\end{pmatrix},$$

$$C_{0.4} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad C_{\mu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $0.4 < \mu \leq 1$. Thus for arbitrary $\lambda \in (0,1]$ and arbitrary F-matrix $D = (d_{ij})_{3\times 3}$, then

$$C_\lambda D_\lambda = D_\lambda C_\lambda = I_3$$

does not hold water.

Therefore C is not a pseudo invertible F-matrix.

Theorem 3 Let $A \in L^{n\times n}$, $n \geq 2$. If A is a invertible F-matrix, then A is a pseudo invertible F-matrix.

Note. Its inverse theorem is untenable. As in example 1 A is a pseudo invertible F-matrix, and A is not invertible, since A is not a permutation matrix.

Corollary 4 If A is a nth order permutation matrix, then A is a pseudo invertible F-matrix.

Theorem 5 (Decision theorem of a pseudo invertible F-matrix) A is a pseudo invertible F-matrix if and only if A is an orthogonal F-matrix.

Proof. $\Leftarrow$ If A is an orthogonal F-matrix, then there is $\lambda \in (0,1]$, so that A_λ is a permutation matrix. Therefore A is a pseudo invertible F-matrix.

$\Rightarrow$ If A is a pseudo invertible F-matrix. Let $A \in L^{n\times n}$, then there is $\lambda \in (0,1]$ and $B \in L^{n\times n}$, so that

$$A_\lambda B_\lambda = B_\lambda A_\lambda = I_n.$$

Since A_λ is a permutation matrix, then A is an orthogonal F-matrix.

By this theorem we know that the definition of a pseudo invertible F-matrix is equivalent to the definition of an orthogonal F-matrix. Therefore, the decision method for an orthogonal F-matrix is also the decision method for pseudo invertible F-matrix.

Theorem 6 If A is a pseudo invertible F-matrix, then the permutation matrix A_λ of A is unique.

For arbitrary pseudo invertible F-matrix $A = (a_{ij})_{n\times n}$

write that

(2) $$\lambda = \min_i(\max_j(a_{ij})),$$

then a pseudo invertible F-matrix of A is $(A_\lambda)'$. Therefore, all pseudo invertible F-matrix of A ought to solve as follow:

First. By formula (2) compute λ.

Second. Write out the permutation matrix A_λ of A.

Third. Write out $(A_\lambda)'$.

Fouth. In $(A_\lambda)'$ exchange $[0,\lambda)$ for 0 and $[\lambda,1]$ for 1. And we get a F-matrix B. And the B is all pseudo invertible F-matrix of A.

Example 3 Solve all pseudo invertible F-matrix of

$$A = \begin{pmatrix} .1 & .1 & .2 & .4 & .2 \\ .7 & .2 & .1 & .3 & .2 \\ .2 & .1 & .4 & .2 & .3 \\ .3 & .2 & .1 & .3 & .6 \\ .1 & .8 & .3 & .3 & .1 \end{pmatrix}$$

By example 3.1 in [1] we know A is a pseudo invertible F-matrix, and

$$\lambda = \min_i(\max_j(a_{ij})) = \min(.7,.8,.4,.4,.6) = 0.4$$

$$A_{0.4} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (A_{0.4})' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Therefore all pseudo invertible F-matrices of A are

$$B = \begin{pmatrix} [0,0.4) & [0.4,1] & [0,0.4) & [0,0.4) & [0,0.4) \\ [0,0.4) & [0,0.4) & [0,0.4) & [0,0.4) & [0.4,1] \\ [0,0.4) & [0,0.4) & [0.4,1] & [0,0.4) & [0,0.4) \\ [0.4,1] & [0,0.4) & [0,0.4) & [0,0.4) & [0,0.4) \\ [0,0.4) & [0,0.4) & [0,0.4) & [0.4,1] & [0,0.4) \end{pmatrix}$$

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