On some spaces of F-numbers sequences

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Abstract. In this paper we introduce some spaces of sequences of fuzzy subsets on R^n which are called F-numbers and show that they are complete metric space.

Keywords. Sequences of F-numbers, bounded convergent sequences of F-numbers, complete metric space.

1. Preliminaries

Let R^n denote n-dimensional Euclidean space, A and B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff metric

$$d_H(A, B) = \mathrm{Max}[\sup_{a\in A} \inf_{b\in B} \| a-b \|, \sup_{b\in B} \inf_{a\in A} \| a-b \|]$$

where $\| . \|$ denotes the usual Euclidean norm in R^n.

Lemma 1 (see [1], theorem 2.1) Let $Q(R^n)$ denote the set of all nonempty, complete subset of R^n. The $(Q(R^n), d_H)$ is a complete metric space.

Definition 1 A fuzzy subset $u: R^n \to [0, 1]$ with the following properties

(a) $\{x\in R^n: u(x) > \alpha\}$ is compact for each $\alpha > 0$

(b) $\{x\in R^n: u(x) = 1\} \neq \varphi$.

is called a F-number.

we denote the set of all F-numbers by $F_0(R^n)$.

Definition 2 A sequence $x = \{x_n\}$ of F-numbers is a function x from the set N of all positive integers into $F_0(R^n)$. The F-number x_n denotes the function at $n \in N$.

Define a map d

$$F_0(R^n) \times F_0(R^n) \to R^1$$

by $$d(u, v) = \sup_{\alpha \geq 0} d_H(L_\alpha(u), L_\alpha(v))$$

where d_H is the Hausdorff metric and we denote by $L_\alpha(u) = \{x\in R^n: u(x) > \alpha\}$, $L_\alpha(v) = \{x\in R^n: v(x) > \alpha\}$.

Lemma 2 (see [1], theorem 4.1) $(F_0(R^n), d)$ is a complete metric space.

We now introduce some spaces of sequences of F-numbers.

$b=\{x=\{x_n\}: \sup_n d(x_n, 0)<\infty\}$
$c=\{x=\{x_n\}:$ there exists $x_0\in F_0(R^n)$ s. t. $d(x_n, 0)\to 0\}$
$c_0=\{x=\{x_n\}: d(x_n, 0)\to 0\}$
$l^p=\{x=\{x_n\}: \sum_n [d(x_n, 0)]^p<\infty\}$ $(1<p<\infty)$
and denote the set of all sequences of F-numbers by S.

2. Main results

Theorem 1 b is a complete metric space with the metric ρ defined by
$$\rho(x, y)=\sup_n d(x_n, y_n)$$
where $x=\{x_n\}$ and $y=\{y_n\}$ are sequences of F-numbers which are in b.

Proof It is straightforward to see that ρ is a metric on b. To show that bis complete in this metric, let $\{x^i\}$ be a Cauchy sequence in b. Then for each fixed n, $\{x_n^i\}$ is a Cauchy sequence in $(F_0(R^n), d)$. But $(F_0(R^n), d)$ is complete, hence, there exsits $x_n\in F_0(R^n)$ such that $\lim_{i\to\infty} x_n^i=x_n$ for every n. Put $x=\{x_n\}$, we shall show that $\lim_{i\to\infty} x^i=x$ and $x\in b$. Since $\{x^i\}$ is a Cauchy sequence in b, given $\varepsilon>0$ there exists $n_0\in N$ such that for $i, j > n_0$ and every $n\in N$, $d(x_n^i, x_n^j)<\varepsilon$

Taking the limit as $j\to\infty$, we get
$$d(x_n^i, x_n)<\varepsilon$$
Therefore $\rho(x^i, x)=\sup_n d(x_n^i, x_n)<\varepsilon$ i. e. $\lim_{i\to\infty} x^i=x$.

It remains to show $x\in b$. From $\rho(x, 0)\leq\rho(x^i, x)+\rho(x^i, 0)<\infty$, we get $\rho(x, 0)=\sup_n d(x_n, 0)<\infty$. That is to say $x=\{x_n\}\in b$ and this proves the completeness of b.

Theorem 2 c is a complete metric space with the metric ρ defined by
$$\rho(x, y)=\sup_n d(x_n, y_n)$$
where $x=\{x_n\}$ and $y=\{y_n\}$ are sequences of F-numbers which are in c_0.

Proof It is clear that (c, ρ) is a metric sepace. To prove the completeness of c, let $\{x^i\}$ be a Cauchy sequence in c. Repeating the proof of theorem 1, we know that there exists $x=\{x_n\}$ such that $\lim_{i\to\infty} x^i=x$. It can be shown by standard arguments that $x\in c$. So the proof is completed.

Theorem 3 c_0 is a complete metric space with the metric ρ defined in the above theorems.

Proof Be similar to the proof of theorem 2.

Theorem 4 l^p is a complete metric space with the metric h defined by
$$h(x, y)=(\sum [d(x_n, y_n)]^p)^{1/p}$$
Where $x=\{x_n\}$ and $y=\{y_n\}$ are sequences of F-numbers which are in l^p.

Proof. Obviously $h(x, y)=0 \iff x=y$ and $h(x, y)=h(y, x)$.

The triangle inequality $h(x, y) < h(x, z) + h(z, y)$ follows from Minkowski inequality

$$(\Sigma_n | a_n+b_n |^p)^{1/p} < (\Sigma_n | a_n |^p)^{1/p} + (\Sigma_n | b_n |^p)^{1/p}$$

and corresponding triangle inequality for d

$$d(x_n, y_n) < d(x_n, z_n) + d(z_n, y_n).$$

Hence, $(1^p, h)$ is a metric space.

To show the completeness of 1^p, let $\{x^i\}$ be a Canchy suquence in 1^p. Then for every fixed n, $\{x_n^i\}$ is a Cauchy sequence in $F_0(R^n)$. Since $(F_0(R^n), d)$ is complete, we have $\lim_{i\to\infty} x_n^i = x_n$ for each n. put $x=\{x_n\}$, it can be proved by standard arguments that $\lim_{i\to\infty} x_n^i = x$ and $x \in 1^p$. So the proof is completed.

Theorem 5 S is a complete metric Space with the metric g defined by

$$g(x, y) = \sum_n \frac{1}{2^n} \frac{d(x_n, y_n)}{1+d(x_n, y_n)}$$

Where $x=\{x_n\}$ and $y=\{y_n\}$ are arbitrary sequences of F-numbers.

Proof. Obviously $g(x, y) > 0$, $g(x, y) = 0 \iff x=y$ and $g(x, y) = g(y, x)$.

The triangle inequality $g(x, y) < g(x, z) + g(z, y)$ follows from that the function $x/(1+x)$ is monotonically increasing and corresponding inequality for d.

Hence, S is a metric space with the metric g.

To prove the completeness of S, let $\{x^i\}$ be a Cauchy Sequence in S. Then given $\varepsilon > 0$, there exists $n_0 \in N$, such that for $i, j > n_0$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d(x_n^i, x_n^j)}{1+d(x_n^i, x_n^j)} < \varepsilon$$

It implies that $\{x_n^i\}$ is a Cauchy sequence in $F_0(R^n)$. By the completeness of $F_0(R^n)$, there exists $x_n \in F_0(R^n)$ such that $\lim_{i\to\infty} x_n^i = x_n$ for every $n \in N$. Put $x=\{x_n\}$, it is easy to show that $\lim_{i\to\infty} x^i = x$. So the completeness of S is proved.

References

[1] M. L. Puri and D. A. Ralescu, Fuzzy random variables, J. Math Anal, Appl 114 (1986) 409-422

[2] M. l. Puri and D. A. Ralescu, Differenticals for fuzzy functions, J. Math Anal. Appl 91 (1983) 552-558