Upper-Continuous And Compactly Generated Properties of Lattice of L-fuzzy Modules

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ABSTRACT: In this note it is shown that the lattice of L-fuzzy modules of a given module is upper-continuous, and under a suitable condition it will be compactly generated.

Keywords: L-fuzzy module, upper-continuous and compactly generated lattice.

1. Introduction

Zadeh in [11] introduced the notion of a fuzzy subset A of a nonempty set X as a function from X to [0,1]. Goguen in [2] generalized the fuzzy subsets of X to L-fuzzy subsets, as a function from X to a lattice L. The concept of fuzzy modules was introduced by Negoita and Ralescu in [7]. Since then several authors have studied fuzzy modules, for example see [3,5,8,9,12]. In [13] it has been shown that the set of all L-fuzzy modules of a given module M over a commutative ring with identity R, has a modular complete, pseudo-complemented lattice structure. Now in this paper firstly it is proved that the lattice of L-fuzzy modules is upper continuous. Then by giving a definition for L-fuzzy module generated by an element of lattice L and a submodule of M, it is given two decomposition theorems for an arbitrary L-fuzzy modules of M. And finally by using these theorems and imposing a suitable condition on L, it will be shown that the lattice of all L-fuzzy modules is compactly generated.

2. Preliminaries

In this note $L = (L, \leq, \sup, \inf)$ stands for a completely distributive lattice which has the least and greatest elements say 0 and 1 respectively. For a nonempty set X, let

$$F(X) = \{A \mid A \text{ is an L-fuzzy subset of } X\}.$$

By an L-fuzzy point x_t for $x \in X$, $t \in L$ we mean $x_t \in F(X)$ which is defined by

$$x_t(y) = \begin{cases} t & \text{if } y = x \\ 0 & \text{if } y \neq x, \end{cases}$$

and we write $x_t \in X$. If x_t is an L-fuzzy point of X and $x_t \leq A \in F(X)$, then we write $x_t \in A$. Let $y \subseteq X$, then χ_y denotes the characteristic function of y, and obviously $\chi_y \in F(X)$.

From now on R is a ring with identity, M is an unitary R-module and all definitions and notations in this paper are follow the presentation of Zahedi [12,13].

Recall that S(M) denotes the set of all L-fuzzy modules of M.

Theorem 2.1 (See Theorem 4.1 of [13]). Let $A, B \in S(M)$. Then

(i) $A \cap B$ is the greatest L-fuzzy module of M, contained in A and B

(ii) $A + B$ is the least L-fuzzy module of M, containing A and B.

Notation 2.2. Let $A, B \in S(M)$. Then by $A \wedge B$ and $A \vee B$ we mean $A \cap B$ and $A + B$ respectively.

Theorem 2.3 (See Theorem 4.3 of [13]). $(S(M), \leq, \wedge, \vee)$ is a modular complete, pseudo-complemented lattice, with the least and greatest elements $\chi_{\langle 0 \rangle}$ and χ_M, respectively.

Notation 2.4 (See [4, page 2]). Let I be a nonempty set, and $x, x_i \in M$ where $i \in I$. By the summation $x = \sum_{i \in I} x_i$ we mean that all but a finite number of the x_α are zero.

Definition 2.5 (See also [4, Definition 1.1]). Let $\{A_i\}_{i \in I}$ be a family of L-fuzzy modules of M. Define the L-fuzzy subset $\sum_{i \in I} A_i$ of M by

$$\left(\sum_{i \in I} A_i\right)(x) = \sup_{x = \sum_{i \in I} x_i} \; \inf_{i \in I} A_i(x_i).$$

3. Main Results

Let $\{A_i\}_{i \in I}$ be a family of L-fuzzy submodules of M. Then as we saw in the proof of Theorem 4.3 of [13], $\bigcap_{i \in I} A_i$

is the greatest lower bound of $\{A_i\}_{i\in I}$ in S(M). Now we show that $\sum_{i\in I} A_i$ is the least upper bound of $\{A_i\}_{i\in I}$.

Theorem 3.1. Let $\{A_i\}_{i\in I}$ be a nonempty subset of S(M). Then $\sum_{i\in I} A_i$ is the least upper bound of $\{A_i\}_{i\in I}$ in the lattice S(M).

Proof. By considering the proof of Proposition 1.2 of [4] and doing similar to that, it is seen that $\sum_{i\in I} A_i$ is an L-fuzzy subgroup of M and $A_i \le \sum_{i\in I} A_i$, for all $i \in I$. And it is easy to check that $(\sum_{i\in I} A_i)(0) = 1$. Now we show that

$$(\sum_{i\in I} A_i)(rx) \ge (\sum_{i\in I} A_i)(x) \quad \text{for all } r \in R,\ x \in M. \qquad (1)$$

We have

$$(\sum_{i\in I} A_i)(rx) = \sup_{rx = \sum_{i\in I} x_i} \ \inf_{i\in I} \ A_i(x_i)$$

$$\ge \sup_{rx = \sum_{i\in I} rx_i} \ \inf_{i\in I} \ A_i(rx_i)$$

$$\ge \sup_{rx = \sum_{i\in I} rx_i} \ \inf_{i\in I}(A_i(x_i)); \text{ since } A_i \in S(M)$$

$$\ge \sup_{x = \sum_{i\in I} x_i} \ \inf_{i\in I}(A_i(x_i))$$

$$= (\sum_{i\in I} A_i)(x).$$

Thus (1) is proved, and hence $(\sum_{i\in I} A_i) \in S(M)$.

Now let $B \in S(M)$ and $A_i \le B$ for all $i \in I$. Then by using Definition 2.5, it is easy to see that $\sum_{i\in I} A_i \le B$. Whence the proof of theorem is completed.

Recall that a complete lattice $(L_o, \le, \wedge, \vee)$ is said to be upper-continuous if for each directed subset A of L_o and any $a \in L_o$ we have $a \wedge (\bigvee_{x\in A} x) = \bigvee_{x\in A} (a \wedge x)$, where a subset A in Lo is said to be directed if for all $a, b \in A$ there is a $c \in A$ such that $a \le c$ and $b \le c$. And an element a in an upper-continuous lattice L_o is said to be compact if for each directed subset A of L_o such that $a \le \bigvee_{x\in A} x$, then there exists an $x_o \in A$ such that $a \le x_o$. An upper-continuous lattice L_o is called compactly generated if each element of L_o is a join of compact elements.

Lemma 3.2. Let $\{A_i\}_{i\in I}$ be a directed family of L-fuzzy modules of M. Then $\sum_{i\in I} A_i = \bigcup_{i\in I} A_i$.

Lemma 3.3. Let $\{A_i\}_{i\in I}$ be a subset of S(M) and $A \in S(M)$. Then

(i) $A \cap (\bigcup_{i\in I} A_i) = \bigcup_{i\in I} (A \cap A_i)$

(ii) if $\{A_i\}_{i\in I}$ is directed then $\{A_i \cap A\}_{i\in I}$ is also directed.

Theorem 3.4. S(M) is an upper continuous lattice.

Definition 3.5. Let $\lambda \in L$, and m be a submodule of M. Then by $\langle\lambda,m\rangle$ we mean an L-fuzzy module of M which is defined by

$$\langle\lambda,m\rangle(x) = \begin{cases} 1 & \text{if } x = 0 \\ \inf(\lambda,\chi_m(x)) & \text{if } x \neq 0, \end{cases}$$

and it is called the L-fuzzy module of M generated by λ and m.

Theorem 3.6 (Generated Decomposition Theorem). Let $A \in S(M)$. Then

$$A = \bigcup_{\lambda \in \operatorname{Im} A} \langle\lambda,A_\lambda\rangle,$$

where $A_\lambda = \{x \in M \mid A(x) \geq \lambda\}$ and Im A is the image of the function A.

Hereafter we let $\mathscr{S}(M)$ denotes the lattice of (ordinary) submodules of M.

Remark 3.7 (i): Let $A \in S(M)$ and $\lambda \in \operatorname{Im} A$. Since A_λ is a submodule of M we get it is a union of a family of finitely generated submodules of itself, say $A_\lambda = \bigcup_{t \in T_\lambda} A_{\lambda,t}$. And as it has been shown in [6, page 45], each of $A_{\lambda,t}$ is a compact element in $\mathscr{S}(M)$.

(ii): Let $m \leq M$ and $m = \bigcup_{\alpha\in\Gamma} m_\alpha$. Then $\chi_m = \bigcup_{\alpha\in\Gamma} \chi_{m\alpha}$, that is $\chi_m(x) = \sup_{\alpha\in\Gamma} \chi_{m\alpha}(x)$.

Now by considering Remark 3.7 we can give the following decomposition theorem.

Theorem 3.8 (Finitely Generated Decomposition Theorem). Let $A \in S(M)$, then

$$A = \bigcup_{\lambda \in \mathrm{Im}\, A,\, t \in T_\lambda} \langle \lambda, A_{\lambda,t} \rangle .$$

Theorem 3.9. Let for any indexed subset $\{a_\alpha\}_{\alpha \in \Lambda}$ of L, there exists an $\alpha_o \in \Lambda$ such that $a_{\alpha_o} = \sup_{\alpha \in \Lambda} a_\alpha$. Then S(M) is compactly generated.

Proof. By considering the definition of compactly generated lattice and Theorem 3.8, it is enough to show that each of $\langle \lambda, A_{\lambda,t} \rangle$ is compact in S(M). To this end, let $\langle \lambda, A_{\lambda,t} \rangle \leq \bigcup_{\alpha \in \Gamma} \mu_\alpha$, where $\{\mu_\alpha\}_{\alpha \in \Gamma}$ is a directed subset of S(M). It is easy to see that $\langle \lambda, A_{\lambda,t} \rangle_\lambda = A_{\lambda,t}$. Thus $A_{\lambda,t} \subseteq (\bigcup_{\alpha \in \Gamma} \mu_\alpha)_\lambda$. Now by using the hypothesis it is seen that

$$A_{\lambda,t} \subseteq (\bigcup_{\alpha \in \Gamma} \mu_\alpha)_\lambda = \bigcup_{\alpha \in \Gamma} (\mu_\alpha)_\lambda . \qquad (5)$$

Because $\{\mu_\alpha\}_{\alpha \in \Gamma}$ is a directed subset of S(M), it can be concluded that $\left\{(\mu_\alpha)_\lambda\right\}_{\alpha \in \Gamma}$ is a directed subset of $\mathcal{S}(M)$. But as it was mentioned in Remark 3.7 (i), every $A_{\lambda,t}$ is compact in $\mathcal{S}(M)$. Thus by (5) we get there exists an $\alpha_o \in \Gamma$ such that $A_{\lambda,t} \subseteq (\mu_{\alpha_o})_\lambda$. This implies that $\langle \lambda, A_{\lambda,t} \rangle \leq \mu_{\alpha_o}$, so $\langle \lambda, A_{\lambda,t} \rangle$ is compact in S(M).

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