Characteristic properties of fuzzy semi-homeomorphic order-homomorphisms

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Abstract: In this paper, first the concepts of semi-bundary-preserving, semi-interior-preserving and semi-closure- preserving ets. in a fuzzy topological space are in troduced and discusced. Then the relation among these are obtained. Finally, We discuss some semi-homeomorphic order-homomorphisms equivalent theorems.

Keywords: fuzzy semi-boundary; semi-boundary-preserving order-homomorphism; semi-homeomorphic order-homomorphism.

Throughout this paper L will denote a fuzzy latlice. Let 0, 1 denote the least element and the greatest element in L respectively. Let δ be a topology on L. Then (L, δ) is called a fuzzy topological space , or briefly fts , The elements of δ are called open elements and the elements of δ' are called closed elements, where $\delta' = \{A' \mid A \in \delta\}$. Let A^o, A^- donote the interior and the closure of A respectively. A elenent $B \in L$, is called a semi-open element of δ, if there exists $O \in \delta$ such that $O \leqslant B \leqslant O^-$ where O^- expresses the closure of O. FSO(L, δ) will denote the family of all semi-open element in (L, δ). If B is semi-open element of δ then B' are called semi-closed element of δ . FSC(L, δ) will denote the family of all semi-closed element in (L, δ) (See [2]).

For each element A in L, let $A_o = \vee\{B \in FSO(L, \delta) \mid B \leqslant A\}$ called the semi-interior of A, and let $A_- = \wedge\{B \in FSC(L, \delta) \mid A \leqslant B\}$ called the semi-closure of A, and let $A_{\Leftrightarrow} = \vee \{a \in M \mid a \leqslant A_- \text{ and } a \not\leqslant A_o\}$ (See[2]) called the fuzzy semi-boundary of A.

Definition 1[2] Let $f: (L_1, \delta_1) \to (L_2, \delta_2)$ be an order

-homomorphism (see[1]). If for each $B \in FSO(L_2, \delta_2)$, $f^{-1}(B) \in FSO(L_1, \delta_1)$, then f is said to be irresolute. If for each $A \in FSO(L_1, \delta_1)$ $(FSC(L_1, \delta_1))$, $f(A) \in FSO(L_2, \delta_2)$ $(FSC(L_2, \delta_2))$, then f is said to be semi-open (semi-closed).

Theorem 1 [2] Let $f: (L_1, \delta_1) \rightarrow (L_2, \delta_2)$ be an order-homomorphism. Then the following conditions are equivalent:

(1) f is irresolute,

(2) For each $C \in FSC(L_2, \delta_2)$, $f^{-1}(C) \in FSC(L_1, \delta_1)$,

(3) For each $A \in L_1$, $f(A_-) \leqslant (f(A))_-$,

(4) For each $B \in L_2$, $f^{-1}(B_o) \leqslant (f^{-1}(B))_o$.

Theorem 2 [2] Let $f: (L_1, \delta_1) \rightarrow (L_2, \delta_2)$ be an order-homomorphism. Then the following conditions are equivalent:

(1) f is irresolute,

(2) For each $A \in L_1$, $f(A_{=}) \leqslant (f(A))_-$,

(3) For each $B \in L_2$, $(f^{-1}(B))_{=} \leqslant f^{-1}(B_-)$.

Theorem 3 [2] Let $f: (L_1, \delta_2) \rightarrow (L_2, \delta_2)$ be an order-homomorphism. Then the following conditions are equivalent:

(1) f is semi-open,

(2) For each $A \in L_1$, $f(A_o) \leqslant (f(A))_o$,

(3) For each $A \in L_1$, $f(A_o) = (f(A_o))_o$.

Theorem 4 [2] Let $f: (L_1, \delta_1) \rightarrow (L_2, \delta_2)$ be an order-homomorphism. Then the following conditian are equivalent:

(1) f is semi-closed,

(2) For each $A \in L_1$, $(f(A))_- \leqslant f(A_-)$,

(3) For each $A \in L_1$, $(f(A_-))_- = f(A_-)$.

Definition 2 Let $f: (L_1, \delta_1) \rightarrow (L_2, \delta_2)$ be an order-homomorphism. Then

(1) f is said to be semi-boundary-irresolute iff for each

$A\in L_1$, we have $f(A_{\ominus})\leqslant(f(A))_{\ominus}$,

(2) f is said to be semi-boundary-closed iff for each $A\in L_1$, we have $(f(A))_{\ominus}\leqslant f(A_{\ominus})$,

(3) f is said to be semi-co-continuous iff for each $A\in L_1$, we have $(f(A))_o\leqslant f(A_o)$.

Theorem 5 Let $f: (L_1, \delta_1)\rightarrow(L_2, \delta_2)$ be an order-homomorphism. Then

(1) if f is semi-boundary-irresolute, then f is irresolute,

(2) if f is semi-boundary-closed, then f is semi-closed.

Proof. (1) Suppose that f is semi-boundary-irresolute. For each $A\in L_1$ it follows that $f(A_{\ominus})\leqslant(f(A))_{\ominus}\leqslant(f(A))_-$ by Definition 2 and Theorem 5 in [2]. Hence f is irresolute from Theorem 2.

(2) Suppose that f is semi-boundary-closed. For each $A\in L_1$, it suffices to show that $(f(A))_-=f(A)\vee(f(A))_{\ominus}\leqslant f(A)\vee f(A_{\ominus})=f(A\vee A_{\ominus})=f(A_-)$ by Theorem 7 in [2] and Definition 2. Hence f is semi-closed from Theorem 4.

Theorem 6 Let $f: (L_1, \delta_1)\rightarrow(L_2, \delta_2)$ be an order-homomorphism and a bijection. Then the following statements are equivalent:

(1) f is irresolute,

(2) f is semi-boundary-irresolute,

(3) f^{-1} is semi-boundary-closed,

(4) f^{-1} is semi-closed,

(5) f^{-1} is semi-open,

(6) f is semi-co-continuous.

Proof. (1) ==> (2) For each $A\in L_1$, if $A_{\ominus}=0$, then it follows that $f(A_{\ominus})\leqslant(f(A))_{\ominus}$. If $A_{\ominus}\neq 0$, then for each $a\in[A]$. We have $a\leqslant A_-$ and $a\not\leqslant A_o$ hence $f(a)\leqslant f(A_-)\leqslant(f(A))_-$ by (1) and Theorem 1, we claim that $f(a)\not\leqslant A_o$. In fact, if not then $f(a)\leqslant(f(A))_o$, sinec f is a bejection, f^{-1} is also an order-homomorphism and $(f^{-1})^{-1}=f$, Hence $(f(A))^o\leqslant((((f(A))\_)')'=((f(A'))\_)\_\leqslant(f((A')\_))'=f(((A')\_)')=f((A)_o)$ so we have $f(a)\leqslant f(A_o)$. This impossible $a\leqslant f^{-1}(f(A_o))=A_o$, but this is impossible otherwise we know that an order-homomorphism maps molecules into molecules so $f(a) \leqslant (f(A))_{\ominus}$ hence $f(A_{\ominus}) \leqslant (f(A))_{\ominus}$.

(2) ==> (3) For each $B \in L_2$ there exists $A \in L_1$ such such that $B=f(A)$ and $A=f^{-1}(B)$. It follows that $f((f^{-1}(B))_{\ominus}) = f(A_{\ominus}) \leqslant (f(A))_{\ominus}$ from (2) and hence $(f^{-1}(B))_{\ominus} \leqslant f^{-1}(B_{\ominus})$.

(3) ==> (4) By theorem 7 in [2] and (3) for each $B \in L_2$, it follows that $f^{-1}(B))_{-} = f^{-1}(B) \vee (f^{-1}(B))_{\ominus} \leqslant f^{-1}(B) \vee f^{-1}(B_{\ominus}) = f^{-1}(B \vee B_{\ominus}) = f^{-1}(B_{-})$.

(4) ==> (5) By (4) and Theorem3 for each $B \in L_2$, we have $f^{-1}(B_o) = f^{-1}((B')_{-}) = (f^{-1}((B')_{-}))' \leqslant ((f^{-1}(B'))_{-})' = ((f^{-1}(B')_{-})' = (f^{-1}(B))_o$. This implies that f^{-1} is semi-open for Theorem 3.

(5) ==> (6) For each $A \in L_1$, let $B=f(A)$, then $A=f^{-1}(B)$. Hence we have $f^{-1}((f(A))_o = f^{-1}(B_o) \leqslant (f^{-1}(B))_o = A_o$ and so $(f(A))_o \leqslant f(A_o)$. Thus f is semi-co-continuous.

(6) ==> (1) For each $B \in L_2$ there exists $A \in L_1$ such that $B=f(A)$ and $A=f^{-1}(B)$, By (6) it follows that $f^{-1}(B_o) = f^{-1}((f(A))_o) \leqslant f^{-1}(f(A_o)) = A_o = (f^{-1}(B))_o$ and hence f is irresolute by theorem 1.

Definition 3 Let $f: (L_1, \delta_1) \to (L_2, \delta_2)$ be an order-homomorphism. then

(1) f is said to be semi-boundary-preserving iff it satisfies the condition $(f(A))_{\ominus} = f(A_{\ominus})$ for each $A \in L_1$.

(2) f is said to be semi-interior-preserving iff it satisfies the condition $(f(A))_o = f(A_o)$ for each $A \in L_1$.

(3) f is said to be semi-closure-preserving iff it ssatisfies the condition $(f(A))_{-} = f(A_{-})$ for each $A \in L_1$.

Theorem 7 Let $f: (L_1, \delta_1) \to (L_2, \delta_2)$ be an order-homomorphism. Then

(1) f is semi-boundary-preserving iff it is semi-boundary-irresolute.

(2) f is semi-interior-preserving iff it is semi-open and semi-co-continuous.

(3) f is semi-closure-preserving iff it is semi-colsed and

irresolute.

proof. It follows immediately from theorem 2-4, definition 3, 4.

Theorem 8 If an order-homomorphism f: $(L_1, \delta_1) \rightarrow (L_2, \delta_2)$ is semi-boundary-preserving, then f is semi-ciosure-preserving.

Proof. It follows immediately from Theorem 5 and Theorem 7.

Theoem 9 Let f: $(L_1, \delta_1) \rightarrow (L_2, \delta_2)$ be an order-homomorphism and a bijection. Then

(1) If f is semi-boundary-preserring than so is f^{-1}.

(2) If f is semi-interior-preserring then so is f^{-1}.

(3) If f is semi-closure-preserring then so is f^{-1}.

Proof. (1) Suppose that f is semi-boundary-preserring. Then for each $B \in L_2$. There exists $A \in L_1$, such that $B=f(A)$ and $A=f^{-1}(B)$, so we have $f((f^{-1}(B))_{\ominus}) = f(A_{\ominus}) = (f(A))_{\ominus} = B_{\ominus}$ and hence $(f^{-1}(B))_{\ominus}=f^{-1}(B_{\ominus})$. Thus f^{-1} is semi-boundary-preserving.

In parallel way we can prove (2) and (3).

Definition 4 Let f: $(L_1, \delta_1) \rightarrow (L_2, \delta_2)$ be an order-homomorphism and a bijection. Then f is said to be semi-homomorphic order-homomorphism if it satisfies the condition: $A \in FSO(L_1, \delta_1)$ iff $f(A) \in FSO(L_2, \delta_2)$.

Theorem 10 Let f: $(L_1, \delta_1) \rightarrow (L_2, \delta_2)$ be an order-homomorphism, and a bijection. Then the following statement equivalent:

(1) f is a semi-homomorphism,

(2) Either f or f^{-1} is semi-boundary-preserving,

(3) Either f or f^{-1} is semi-interior-preserving,

(4) Either f or f^{-1} is semi-clousure-preserving,

(5) Either f or f^{-1} is irresolute and Semi-open,

(6) Either f or f^{-1} is irresolute and Semi-closed,

(7) Either f or f^{-1} is semi-co-cotinuous and semi-open,

(8) Either for f^{-1} is semi-co-continuous and semi-closed,

(9) Either f or f^{-1} is semi-boundary-irresolute and semi-open,

(10) Either f or f^{-1} is semi-bounddary-irresolute and semi-closed,

(11) Either f or f^{-1} is irresolute and semi-boundary-closed ,

(12) Either f or f^{-1} is semi-boundary-irresolute and semi-boundary-closed,

(13) Either f or f^{-1} is semi-co-continouns and semi-boundary-closed,

(14) Both f and f^{-1} are semi-open,

(15) Both f and f^{-1} are semi-closed,

(16) Both f and f^{-1} are semi-boundary-cloused,

(17) Both f and f^{-1} are semi-boundaary-irresolute,

(18) Both f and f^{-1} are semi-co-continuous,

(19) Either f or f^{-1} is semi-open and the other is semi-closed,

(20) Either f or f^{-1} is semi-open and the other is semi-boundary-closed,

(21) Either f or f^{-1} is semi-closed and the other is semi-boundary-closed,

(22) Eithey f or f^{-1} is irresolute and the other is semi-co-continous,

(23) Either f or f^{-1} is irresolute and the other is semi-boundary-irresolute,

(24) Either f or f^{-1} is semi-co-continuous and the other is semi-boundary-irresolute.

Proof. The equivalence of (1) - (24) can be shown from theorem 6-9.

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