Fuzzy Algebraic Extension Generated By A Fuzzy Subset And Maximal Fuzzy Algebraic Extension

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Abstract

In this paper first it is shown that if a fuzzy subset is algebraic over a fuzzy field of a given field then the fuzzy field generated by that fuzzy subset is itself algebraic. Then the maximal fuzzy intermediate subfields which are respectively algebraic, purely inseparable and seperable algebraic are constructed.

Keywords: Fuzzy (algebraic) field extension, fuzzy purely inseparable and fuzzy separable algebraic field extensions.

1. Introduction

The concept of fuzzy subfield was introduced by Nanda [5]. Mordeson [3] defined the concepts of fuzzy algebraic, purely inseparable and separable algebraic field extensions, and obtained some interesting results in [3,4]. Also he proved the existence of the following maximal fuzzy subfields under some conditions, purely inseparable and separable algebraic [3, Proposition 2.4, Theorem 2.8].

In this paper we follow Mordeson and give some related results about the above mentioned concepts. In this regards first it is shown that the fuzzy subfield generated by a fuzzy subset algebraic over a given fuzzy subfield which satisfies the sup property condition, is itself algebraic (Theorem 3.5). Then the maximal intermediate subfields

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which are respectively algebraic, purely inseparable and separable algebraic are exactly constructed (Theorem 4.8).

2. Preliminaries

In this paper F denotes a field with characteristic $p > 0$, unless otherwise is stated.

Let $x \in F$, $r \in [0,1]$. Then by a fuzzy point x_r of F, which is denoted by $x_r \in F$, we mean a fuzzy subset of F, which is defined by

$$x_r(y) = \begin{cases} r, & \text{if } y = x \\ 0, & \text{otherwise.} \end{cases}$$

If $x_r \in F$ and A be a fuzzy subset of F such that $x_r \subseteq A$, then we write $x_r \in A$. The addition and multiplication of two fuzzy points x_r and y_s are given by

$$x_r + y_s = (x + y)_{\min(r,s)}$$

$$x_r \cdot y_s = (x \cdot y)_{\min(r,s)}$$

Definition 2.1. Let A be a fuzzy subset of F. Then A is called a fuzzy subfield iff

(i) $A(x-y) \geq \min \{A(x), A(y)\}$

(ii) $A(x \cdot y^{-1}) \geq \min \{A(x), A(y)\}$, $y \neq 0$

(iii) $A(1) = 1$

From now on A and B denote then fuzzy subfields of F.

Definition 2.2. The product of A and B is defined by

$$(AB)(x) = \sup_{x = \sum_{i=1}^{n} y_i z_i,\ n \in \mathbb{N}} \{\min \{\min \{A(y_i), B(z_i)\} \mid i = 1,2,\ldots,n\}\}.$$

Notation: If $B \subseteq A$, then we write A/B and call A/B is a fuzzy field extension.

Definition 2.3. Let A/B be a fuzzy field extension and C a fuzzy subfield of F such that $B \subseteq C \subseteq A$. Then C is called a fuzzy intermediate field of A/B.

Let $\xi(A/B)$ denotes the set of all fuzzy intermediate fields of A/B.

Definition 2.4. Let $c \in F$ and $t \in [0,1]$. Then

(i) c_t is said to be fuzzy algebraic over B if and only if there exists $n \in \mathbb{N}$, $k_i \in F$, and $\lambda_i \in [0,1]$ with $(k_i)_{\lambda_i} \in B$ for i=1,...,n and $k_n \neq 0$ such that

$$k_{n\lambda_n}(c_t)^n + \ldots + k_{1\lambda_1}c_t + k_{0\lambda_0} = 0_t .$$

(ii) c_t is said to be fuzzy purely inseparable over B if and only if there exists a nonnegative integer e such that $(c_t)^{p^e} \in B$.

(iii) c_t is said to be fuzzy separable algebraic over B if and only if for all $D \in \xi(A/B)$, either $D(c) \geq t$ or $D(c) = D(c^P)$.

Definition 2.5. Let X be a fuzzy subset of F. Then X is called fuzzy algebraic over B if and only if x_t is fuzzy algebraic over B for all $x_t \in X$.

Definition 2.6. A/B is called fuzzy algebraic extension if and only if A is fuzzy algebraic over B.

Definition 2.7. A/B is called fuzzy purely inseparable (separable algebraic) extension iff c_t is fuzzy purely inseparable (separable algebraic) over B, for all $c_t \in A$.

3. Fuzzy algebraic extension generated by a fuzzy subset

In this section we let X be a fuzzy subset of F such that X(1) = 1. And it is not necessary that the characteristic of F be prime.

Lemma 3.1. (i) Let $x_t \in X$. Then x_t is fuzzy algebraic over B if and only if x is algebraic over B_t.

(ii) If X is fuzzy algebraic over B, then X_t is algebraic over B_t for all $t \in [0,1]$.

(iii) If X_t is algebraic over B_t for all $t \in Im(X)$, then X is fuzzy algebraic over B.

Definition 3.2. By a fuzzy field extension generated by B and X, which is denoted by B(X), we mean $B(X) = \bigcap_{C} C$, where the intersection is taken over all fuzzy subfields of F containing B and X.

Theorem 3.3. Let B and X satisfy the sup property condition. Then

$$(B(X))_t = B_t(X_t), \text{ for all } t \in [0,1].$$

Remark 3.4. Although Mordeson in [4, Proposition 2.4] has proved a similar results, but here we replac^ the sup property condition on B and X instead of B(X) which is used in [4]. And our proof is different form it.

Theorem 3.5. Let A = B(X), and B, X satisfy the sup property condition. If X is fuzzy algebraic over B, then A/B is a fuzzy algebraic extension.

Theorem 3.6. Let B_1, B_2 be fuzzy subfields of F and B_1, B_2 and X satisfy the sup property condition. If B_1 is fuzzy algebraic over B_2, then $B_1(X)$ is fuzzy algebraic over $B_2(X)$.

4. Maximal fuzzy algebraic (purely inseparable, separable algebraic) intermediate field

Notation. From now on we let $B \subseteq A$, and $\varphi = Im(A) \cup Im(B)$.

Lemma 4.1. (i) [3,Proposition 1.7] Suppose $c_t \in A$. c_t is fuzzy purely inseparable over B if and only if c is purely inseparable over B_t.

(ii) [3,Theorem 1.8] A/B is fuzzy purely inseparable if and only if A_t/B_t is purely inseparable for all $t \in Im(A)$.

Lemma 4.2. (i) [3,Proposition 1.13] Suppose $c_t \in A$. If c_t is separable algebraic over B_t, then c_t is fuzzy separable algebraic over B.

Conversely, if B has the sup property and c_t is fuzzy separable algebraic over B, then c is separable algebraic over B_t.

(ii) [3,Theorem 1.14] If A_t/B_t is separable algebraic for all $t \in Im(A)$, then A/B is fuzzy separable algebraic. Conversely, if B has the sup property and A/B is fuzzy separable algebraic, then A_t/B_t is separable algebraic for all $t \in Im(A)$.

Lemma 4.3. Let $t,s \in [0,1]$, $t \le s$, $c_s \in A$ and c_s is fuzzy purely inseparable over B. Then c_t is also fuzzy purely inseparable over B .

Lemma 4.4. Let B satisfies the sup property condition and $c_t \in A$. Then $c_t \in B$ if and only if c_t is fuzzy purely inseparable and fuzzy separable algebraic over B .

Lemma 4.5. Let $x \in F$ and

$$t_x = \sup \{ t \in Im(A) \mid x \in A_t \}.$$

Then $t_x = A(x)$. Moreover if $t \in [0,1]$ and $t > t_x$, then $x \notin A_t$.

Lemma 4.6. Let $x \in F$ and

$$t'_x = \sup \{ t \in Im(B) \mid x \text{ is algebraic over } B_t \}.$$

Then x is not algebraic over B_t for any $t > t'_x$, where $t \in [0,1]$.

Lemma 4.7. Let $x \in F$ and

$$t'_x = \sup \{ t \in Im(B) \mid x \text{ is purely inseparable (separable algebraic) over } B_t \}.$$

then x is not purely inseparable (separable algebraic) over B_t, for any $t > t'_x$, where $t \in [0,1]$

In the following A and B satisfy the sup property condition.

Theorem 4.8. (i) Let $E = \bigcup_{c_t \in A} c_t$ where $t \in \varphi$, and c_t is fuzzy algebraic over B. Then E is a fuzzy subfield of F and E/B is a fuzzy algebraic extension.

(ii) Let A/B be a fuzzy algebraic extension and $P = \bigcup_{c_t \in A} c_t$ where $t \in \varphi$, and c_t is fuzzy purely inseparable over B. Then P is a fuzzy subfield of F and P/B is a fuzzy purely inseparable extension.

(iii) Let A/B be a fuzzy algebraic extension and $S = \bigcup_{c_t \in A} c_t$ where $t \in \varphi$, and c_t is fuzzy separable algebraic over B. Then S is a fuzzy subfield of F and S/B is a fuzzy separable algebraic extension.

Hereafter E, S and P are defined as in Theorem 4.8.

Theorem 4.9. (i) Let $E' = \bigcup_{c_t \in A} c_t$ where $t \in [0,1]$ and c_t is fuzzy algebraic over B, then $E' = E$.

(ii) Let $P' = \bigcup_{c_t \in A} c_t$ where $t \in [0,1]$ and c_t is fuzzy purely inseparable over B, then $P' = P$.

(iii) Let $S' = \bigcup_{c_t \in A} c_t$ where $t \in [0,1]$ and c_t is fuzzy separable algebraic over B, then $S' = S$.

Theorem 4.10. E, S and P are maximal fuzzy intermediate subfields of $\xi(A/B)$, which are respectively fuzzy algebraic, fuzzy separable algebraic and fuzzy purely inseparable over B.

Theorem 4.11. (i) $P \cap S = B$,

(ii) A/S is a fuzzy purely inseparable extension,

(iii) A/P is a fuzzy separable algebraic extension if and only if $A = SP$.

Theorem 4.12. Let $G \in \xi(A/B)$ and G satisfies the sup property condition. Then the following statements hold,

(i) A/G is fuzzy purely inseparable extension if and only if $S \subseteq G$,

(ii) If A/G is fuzzy separable algebraic extension, then $P \subseteq G$,

(iii) If $S \cap G = B$, then $G \subseteq P$.

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