NORMS AND METRICS OVER INTUITIONISTIC FUZZY SETS

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Let a set E be fixed. An IFS A in E is an object having the form (see [1]):

$$A = \{<x, \mu_A(x), \gamma_A(x)>/x\in E\},$$

where the functions $\mu_A : E \to [0, 1]$ and $\gamma_A : E \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x\in E$ to the set A, which is a subset of E, respectively, and for every $x\in E$:

$$0 \le \mu_A(x) + \gamma_A(x) \le 1.$$

Obviously, every ordinary fuzzy set has the form:

$$\{<x, \mu_A(x), 1-\mu_A(x)>/x\in E\}.$$

For every two IFSs A and B are valid (see [1-3]) the following definitions (let $\alpha, \beta \in [0, 1]$):

$A \subset B$ iff $(\forall x\in E)(\mu_A(x) \le \mu_B(x)$ & $\gamma_A(x) \ge \gamma_B(x))$;

$A = B$ iff $(\forall x\in E)(\mu_A(x) = \mu_B(x)$ & $\gamma_A(x) = \gamma_B(x))$;

$\bar{A} = \{<x, \gamma_A(x), \mu_A(x)>/x\in E\}$;

$A \cap B = \{<x, \min(\mu_A(x), \mu_B(x)), \max(\gamma_A(x), \gamma_B(x))>/x\in E\}$;

$A \cup B = \{<x, \max(\mu_A(x), \mu_B(x)), \min(\gamma_A(x), \gamma_B(x))>/x\in E\}$;

$A + B = \{<x, \mu_A(x)+\mu_B(x)-\mu_A(x).\mu_B(x), \gamma_A(x).\gamma_B(x)>/x\in E\}$;

$A . B = \{<x, \mu_A(x).\mu_B(x), \gamma_A(x)+\gamma_B(x)-\gamma_A(x).\gamma_B(x)>/x\in E\}$;

$A @ B = \{<x, (\mu_A(x)+\mu_B(x))/2, (\gamma_A(x)+\gamma_B(x))/2>/x\in E\}$;

$\Box A = \{<x, \mu_A(x), 1-\mu_A(x)>/x\in E\}$;

$\Diamond A = \{<x, 1-\gamma_A(x), \gamma_A(x)>/x\in E\}$;

$C(A) = \{<x, K, L>/x\in E\}$, where $K = \max_{x\in E} \mu_A(x)$, $L = \min_{x\in E} \gamma_A(x)$;

$I(A) = \{<x, k, l>/x\in E\}$, where $k = \min_{x\in E} \mu_A(x)$, $l = \max_{x\in E} \gamma_A(x)$;

$D_\alpha(A) = \{<x, \mu_A(x)+\alpha.\pi_A(x), \gamma_A(x)+(1-\alpha).\pi_A(x)>/x\in E\}$;

$F_{\alpha,\beta}(A) = \{<x, \mu_A(x)+\alpha.\pi_A(x), \gamma_A(x)+\beta.\pi_A(x)>/x\in E\}$, where $\alpha+\beta \le 1$;

$$G_{\alpha,\beta}(A) = \{<x, \alpha.\mu_A(x), \beta.\gamma_A(x)>/x\in E\};$$

$$H_{\alpha,\beta}(A) = \{<x, \alpha.\mu_A(x), \gamma_A(x)+\beta.\pi_A(x)>/x \in E\};$$

$$H^*_{\alpha,\beta}(A) = \{<x, \alpha.\mu_A(x), \gamma_A(x)+\beta.(1-\alpha.\mu_A(x)-\gamma_A(x))>/x \in E\};$$

$$J_{\alpha,\beta}(A) = \{<x, \mu_A(x)+\alpha.\pi_A(x), \beta.\gamma_A(x)>/x \in E\};$$

$$J^*_{\alpha,\beta}(A) = \{<x, \mu_A(x)+\alpha.(1-\mu_A(x)-\beta.\gamma_A(x)), \beta.\gamma_A(x)>/x \in E\}.$$

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Let us emphasize in the first place that here we do not study these properties of the IFSs which follow directly from the fact that IFSs are sets in the sense of the set theory (see [1]). For example, when E is a metric space, one can study the metric properties of the IFSs over E. This can be made directly by topological methods (see e.g. [4]), and the essential properties of the IFSs are not used. On the other hand, all IFSs (respectively - all fuzzy sets) over a fixed universe E generate a metric space (in the sense of [4, 5]), but with a special metric: a metric, which is not related to the elements of E and to the values of the functions μ_A and γ_A, defined for these elements.

This peculiarity is based on the fact, that the "norm" of a given IFS's element is really not a norm in the sense of [4, 5]. This "norm" is in some sense a "pseudo-norm", which assigns a number to the element $x\in E$. This number is related with the values of the functions μ_A and γ_A (which are calculated for this element). Thus the important conditions:

$$\|x\| = 0 \text{ iff } x = 0,$$
$$\|x\| = \|y\| \text{ iff } x = y,$$

are not valid here. Instead the following ones are valid:

$$\|x\| = \|y\| \text{ iff } \mu_A(x) = \mu_A(y) \ \& \ \gamma_A(x) = \gamma_A(y).$$

Really, the value of $\mu_A(x)$ plays the role of a norm (more precise - pseudo-norm) for the element $x \in E$ in every fuzzy set over E. In the case of the intuitionistic fuzzing, the existing of the second functional component - the function γ_A - generates different possibilities for the defining of the concept "norm" (in the sense of pseudo-norm) over the subsets and over the elements of a given universe E.

The first norm for every $x \in E$ about a fixed set $A \subset E$ is:

$$\sigma_A(x) = \|x\|_1^A = \mu_A(x) + \gamma_A(x).$$

It assigns the degree of "definiteness" of the element x. From

$$\pi(x) = 1 - \mu(x) - \gamma(x)$$

it follows that

$$\sigma_A(x) = 1 - \pi_A(x).$$

THEOREM 1: For every two IFSs A and B, for every $x \in E$ and for every two $\alpha, \beta \in [0, 1]$:

(a) $\sigma_{\bar{A}}(x) = \sigma_{\bar{A}}(x)$;

(b) $\sigma_{A \cap B}(x) \geq \min(\sigma_A(x), \sigma_B(x))$;

(c) $\sigma_{A \cup B}(x) \leq \max(\sigma_A(x), \sigma_B(x))$;

(d) $\min(\sigma_A(x), \sigma_B(x)) \leq \sigma_{A-B}(x) \leq \max(\sigma_A(x), \sigma_B(x))$;

(e) $\sigma_{A+B}(x) \geq \sigma_A(x).\sigma_B(x)$;

(f) $\sigma_{A.B}(x) \geq \sigma_A(x).\sigma_B(x)$;

(g) $\sigma_{A@B}(x) = (\sigma_A(x) + \sigma_B(x))/2$;

(h) $\sigma_{\Box A}(x) = 1$;

(i) $\sigma_{\Diamond A}(x) = 1$;

(j) $\sigma_{CA}(x) \geq \max_{x \in E} \sigma_A(x)$;

(k) $\sigma_{IA}(x) \leq \min_{x \in E} \sigma_A(x)$;

(l) $\sigma_{D_\alpha(A)}(x) = 1$;

(m) $\sigma_{F_{\alpha,\beta}(A)}(x) = \alpha + \beta + (1 - \alpha - \beta).\sigma_A(x)$, for $\alpha+\beta \leq 1$;

(n) $\sigma_{G_{\alpha,\beta}(A)}(x) \leq \sigma_A(x)$;

(o) $\sigma_{H_{\alpha,\beta}(A)}(x) \leq \beta + (\alpha+\beta).\sigma_A(x)$;

(p) $\sigma_{H^*_{\alpha,\beta}(A)}(x) \leq \beta + (1-\beta).\sigma_A(x)$;

(q) $\sigma_{J_{\alpha,\beta}(A)}(x) \leq \alpha + (\alpha+\beta).\sigma_A(x)$;

(r) $\sigma_{J^*_{\alpha,\beta}(A)}(x) \leq \alpha + (1-\alpha).\sigma_A(x)$;

(s) $\sigma_{!A}(x) \geq 0$;

(t) $\sigma_{?A}(x) \geq 0$;

(u) $\sigma_{K_\alpha(A)}(x) \geq 0$;

(v) $\sigma_{L_\alpha(A)}(x) \geq 0$;

(w) $\sigma_{P_{\alpha,\beta}(A)}(x) \geq 0$, for $\alpha+\beta \leq 1$;

(x) $\sigma_{Q_{\alpha,\beta}(A)}(x) \geq 0$, for $\alpha+\beta \leq 1$.

THEOREM 2: For every two IFS A and B and for every $x \in E$, if

$$A = B, \text{ then } \sigma_A(x) = \sigma_B(x).$$

The second norm, which we shall define for every $x \in E$ about a fixed $A \subset E$ is:

$$\partial_A(x) = (\mu_A(x)^2 + \gamma_A(x)^2)^{1/2}$$

Thus defined the two norms are analogous to both basic classical types of norms.

For the norm ∂ the following assertions are valid.

THEOREM 1: For every two IFSs A and B, for every $x \in E$ and for every two $\alpha, \beta \in [0, 1]$:

(a) $\partial_{\bar{A}}(x) = \partial_A(x)$;

(b) $\partial_{A\cap B}(x) \geq \min(\partial_A(x), \partial_B(x))$;

(c) $\partial_{A\cup B}(x) \leq \max(\partial_A(x), \partial_B(x))$;

(d) $\min(\partial_A(x), \partial_{\bar{B}}(x)) \leq \partial_{A-B}(x) \leq \max(\partial_A(x), \partial_{\bar{B}}(x))$;

(e) $\partial_{A+B}(x) \geq \partial_A(x).\partial_B(x)$;

(f) $\partial_{A.B}(x) \geq \partial_A(x).\partial_B(x)$;

(g) $\partial_{A@B}(x) \leq 1/\sqrt{2}.(\partial_A + \partial_B)$;

(h) $\partial_{\Box A}(x) \geq 1 - \mu_A(x)$;

(i) $\partial_{\Diamond A}(x) \geq 1 - \gamma_A(x)$;

(j) $\partial_{CA}(x) \leq \max_{x \in E} \partial_A(x)$;

(k) $\partial_{IA}(x) \geq \min\limits_{x \in E} \partial_A(x)$;

(l) $\partial_{D_\alpha(A)}(x) \geq \partial_A(x)$;

(m) $\partial_{F_{\alpha,\beta}(A)}(x) \geq \partial_A(x)$, for $\alpha + \beta \leq 1$;

(n) $\partial_{G_{\alpha,\beta}(A)}(x) \leq \partial_A(x)$;

(o) $\partial_{H_{\alpha,\beta}(A)}(x) \geq \alpha.\partial_A(x)$;

(p) $\partial_{H^*_{\alpha,\beta}(A)}(x) \geq \alpha.\partial_A(x)$;

(q) $\partial_{J_{\alpha,\beta}(A)}(x) \geq \beta.\partial_A(x)$;

(r) $\partial_{J^*_{\alpha,\beta}(A)}(x) \geq \beta.\partial_A(x)$;

(s) $\partial_{!A}(x) \geq 1/\sqrt{2}$;

(t) $\partial_{?A}(x) \geq 1/\sqrt{2}$;

(u) $\partial_{K_\alpha(A)}(x) \geq 1/\sqrt{\alpha}$;

(v) $\partial_{L_\alpha(A)}(x) \geq 1/\sqrt{\alpha}$;

(w) $\partial_{P_{\alpha,\beta}(A)}(x) \geq 1/\sqrt{\alpha}$, for $\alpha + \beta \leq 1$;

(x) $\partial_{Q_{\alpha,\beta}(A)}(x) \geq 1/\sqrt{\alpha}$, for $\alpha + \beta \leq 1$.

THEOREM 4: For every two IFS A and B and for every $x \in E$, if

$$A = B, \text{ then } \partial_A(x) = \partial_B(x).$$

We note that from $A \subset B$ it does not follow that

$$\partial_A(x) \leq \partial_B(x).$$

For example, if

$$E = \{x\},\ A = \{\langle x, 0.3, 0.4\rangle\},\ B = \{\langle x, 0.4, 0.09\rangle\},$$

then $A \subset B$, but

$$\partial_A(x) = 0.5 > 0.41 = \partial_B(x)$$

This is also valid for the first norm.

For a given finite universe E and for a given IFS A the following discrete norms can be defined:

$$n_\mu(A) = \sum_{x\in E} \mu_A(x),$$

$$n_\gamma(A) = \sum_{x\in E} \gamma_A(x),$$

$$n_\pi(A) = \sum_{x\in E} \pi_A(x),$$

which can be extended to continuous norms. In this case the sum Σ is changed with an integral over E. The above norms can be normalized on the interval [0, 1]:

$$n^*_\mu(A) = (\sum_{x\in E} \mu_A(x))/card(E),$$

$$n^*_\gamma(A) = (\sum_{x\in E} \gamma_A(x))/card(E),$$

$$n^*_\pi(A) = (\sum_{x\in E} \pi_A(x))/card(E),$$

where card(E) is the cardinality of the set E.

THEOREM 5: For every two IFSs A and B and for every two $\alpha, \beta \in [0, 1]$:

(a) $n_\mu(\bar{A}) = n_\gamma(A)$;

(b) $n_\gamma(\bar{A}) = n_\mu(A)$;

(c) $n_\mu(A\cap B) \le \min(n_\mu(A), n_\mu(B))$;

(d) $n_\gamma(A\cap B) \ge \max(n_\gamma(A), n_\gamma(B))$;

(e) $n_\mu(A\cup B) \ge \max(n_\mu(A), n_\mu(B))$;

(f) $n_\gamma(A\cup B) \le \min(n_\gamma(A), n_\gamma(B))$;

(g) $n_\mu(A+B) \le n_\mu(A) + n_\mu(B))$;

(h) $n_\gamma(A+B) \le \min(n_\gamma(A), n_\gamma(B))$;

(i) $n_\mu(A.B) \le \min(n_\mu(A), n_\mu(B))$;

(j) $n_\gamma(A.B) \le n_\gamma(A) + n_\gamma(B))$;

(k) $n_\mu(A@B) = (n_\mu(A), n_\mu(B))/2$;

(l) $n_\gamma(A@B) = (n_\gamma(A), n_\gamma(B))/2$;

(m) $n_\mu(\Box A) = n_\mu(A)$;

(n) $n_\gamma(\Box A) = 1 - n_\mu(A)$;

(o) $n_\mu(\Diamond A) = 1 - n_\mu(A)$;

(p) $n_\gamma(\Diamond A) = n_\gamma(A)$;

(q) $n_\mu(CA) \leq card(E)$;

(r) $n_\gamma(CA) \leq card(E)$;

(s) $n_\mu(IA) \leq card(E)$;

(t) $n_\gamma(IA) \leq card(E)$;

(u) $n_\mu(D_\alpha(A)) = n_\alpha(A) + \alpha.n_\pi(A)$;

(v) $n_\gamma(D_\alpha(A)) = n_\gamma(A) + (1-\alpha).n_\pi(A)$;

(w) $n_\mu(F_{\alpha,\beta}(A)) = n_\mu(A) + \alpha.n_\pi(A)$, for $\alpha + \beta \leq 1$;

(x) $n_\mu(F_{\alpha,\beta}(A)) = n_\gamma(A) + \beta.n_\pi(A)$, for $\alpha + \beta \leq 1$;

(y) $n_\mu(G_{\alpha,\beta}(A)) = \alpha.n_\mu(A)$;

(z) $n_\gamma(G_{\alpha,\beta}(A)) = \beta.n_\gamma(A)$;

(A) $n_\mu(H_{\alpha,\beta}(A)) = \alpha.n_\mu(A)$;

(B) $n_\gamma(H_{\alpha,\beta}(A)) = n_\gamma(A) + \beta.n_\pi(A)$;

(C) $n_\mu(H^*_{\alpha,\beta}(A)) = \alpha.n_\mu(A)$;

(D) $n_\gamma(H^*_{\alpha,\beta}(A)) = n_\gamma(A) + \beta.(1 - \alpha.n_\mu(A) - n_\gamma(A))$;

(E) $n_\mu(J_{\alpha,\beta}(A)) = n_\mu(A) + \alpha.n_\pi(A)$;

(F) $n_\gamma(J_{\alpha,\beta}(A)) = \beta.n_\gamma(A)$;

(G) $n_\mu(J^*_{\alpha,\beta}(A)) = n_\mu(A) + \alpha.(1 - n_\mu(A) - \beta.n_\gamma(A))$;

(H) $n_\gamma(J^*_{\alpha,\beta}(A)) = \beta.n_\gamma(A)$;

(I) $n_\mu(!A) \geq 1/2.card(E)$;

(J) $n_\gamma(!A) \leq 1/2.card(E)$;

(K) $n_\mu(?A) \leq 1/2.card(E)$;

(L) $n_\gamma(?A) \geq 1/2.card(E)$;

(M) $n_{\mu}(K_{\alpha}(A)) \ge \alpha.card(E)$;

(N) $n_{\gamma}(K_{\alpha}(A)) \le \alpha.card(E)$;

(O) $n_{\mu}(L_{\alpha}(A)) \le \alpha.card(E)$;

(P) $n_{\gamma}(L_{\alpha}(A)) \ge \alpha.card(E)$;

(Q) $n_{\mu}(P_{\alpha,\beta}(A)) \ge \alpha.card(E)$, for $\alpha + \beta \le 1$;

(R) $n_{\gamma}(P_{\alpha,\beta}(A)) \le \beta.card(E)$, for $\alpha + \beta \le 1$;

(S) $n_{\mu}(Q_{\alpha,\beta}(A)) \le \alpha.card(E)$, for $\alpha + \beta \le 1$;

(T) $n_{\gamma}(Q_{\alpha,\beta}(A)) \ge \beta.card(E)$, for $\alpha + \beta \le 1$.

For the functions n^*_{μ}, n^*_{γ} and n^*_{π} assertions are valid which are analogous to the above ones (the norm n^*_{π} is defined as the other two norms).

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In the theory of fuzzy sets (see e.g. [6]) two different types of distances are defined, generated from the following metric

$$m_A(x, y) = |\mu_A(x) - \mu_A(y)|.$$

Here the Hemming's and Euclid's metrics coincide. In the case of the intuitionistic fuzziness these metrixes are different:

$$h_A(x, y) = \frac{1}{2}.(|\mu_A(x) - \mu_A(y)| + |\gamma_A(x) - \gamma_A(y)|)$$

(Heming's metrix) and

$$e_A(x, y) = (\frac{1}{2}.((\mu_A(x) - \mu_A(y))^2 + (\gamma_A(x) - \gamma_A(y))^2))^{1/2}$$

(Euclid's metrix).

When the equality

$$\gamma_A(x) = 1 - \mu_A(x),$$

is valid, both metrixes are reduced to the metrix $m_A(x, y)$.

For proving, that h_A and e_A are pseudo-metrixes over E in the sense of [4], we must prove that for every three elements x, y, z $\in$ E:

$$h_A(x, y) + h_A(y, z) \ge h_A(x, z),$$

$$h_A(x, y) = h_A(y, x),$$

$$e_A(x, y) + e_A(y, z) \geq e_A(x, z),$$

$$e_A(x, y) = e_A(y, x).$$

The third equality, which characterizes the metrics (as above) is not valid. Therefore h_A and e_A are pseudo-mertrixes. The proofs of the above four equalities and inequalities are trivial.

The two types distances, defined for the fuzzy sets A and B are:

$$H(A, B) = \sum_{x \in E} |\mu_A(x) - \mu_B(x)|$$

(Hemming's distance) and

$$E(A, B) = (\sum_{x \in E} (\mu_A(x) - \mu_B(x))^2)^{1/2}$$

(Euclid's distance).

These distances, transformed over the IFSs, have the respective forms:

$$H(A, B) = \frac{1}{2}. \sum_{x \in E} |\mu_A(x) - \mu_B(x)| + |\gamma_A(x) - \gamma_B(x)|$$

and

$$E(A, B) = (\frac{1}{2}. \sum_{x \in E} (\mu_A(x) - \mu_B(x))^2 + (\gamma_A(x) - \gamma_B(x))^2)^{1/2}.$$

We shall note, that:

$$H(A, \bar{A}) = \sum_{x \in E} |\mu_A(x) - \gamma_A(x)|$$

and

$$E(A, \bar{A}) = (\sum_{x \in E} (\mu_A(x) - \gamma_A(x))^2)^{1/2}$$

Other distances (cf. [4]), which can be defined over the IFSs are:

$$J_1(A, B) = \max_{x \in E} |\mu_A(x) - \mu_B(x)|$$

$$J_2(A, B) = \max_{x \in E} |\gamma_A(x) - \gamma_B(x)|$$

$$J(A, B) = \frac{1}{2}.(J_1(A, B) + J_2(A, B))$$

$$J^*(A, B) = \frac{1}{2}.\max_{x \in E}(|\mu_A(x) - \mu_B(x)| + |\gamma_A(x) - \gamma_B(x)|)$$

Obviously, for every two IFSs A and B:

$$J^*(A, B) \leq J_1(A, B) + J_2(A, B).$$

Let us note, that the number:

$$|\mu_A(x) - \mu_B(x)| + |\gamma_A(x) - \gamma_B(x)|$$

can be larger than 1. The distances J_1 and J_2 characterize only the components μ and γ and obviously J_1 is reduced directly to the distance for fuzzy sets, while for J_2, J and J^* - this is not valid.

Finally, we shall note that will be very important to define other norms and metrices in the sense of the results e.g. of [7, 8].

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