Fuzzy semigroups and level subsemigroups

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Abstract: In this paper, we discuss the relationship between a given fuzzy semigroup and its level subsemigroups, and investigate the conditions under which a given semigroup has a properly inclusive chain of subsemigroups. In particular, we formulate how to structure a fuzzy semigroup by a given chain of subsemigroups.

Key words: Fuzzy semigroup; Level subsemigroup; Chain of subsemigroups

1. Preliminaries

Zadeh, along with the notion of a fuzzy set in his classical paper [1] also introduced the concept of level subsets. In [2], Rosenfeld demonstrated the utility of the notion of level subsets of a fuzzy set by establishing that the level subsets of a fuzzy subgroupoid or fuzzy (left, right) ideal are subgroupoids or(left, right) ideals in the usual algebraic sense. The idea of level subset was later used by Das in [3] and Dixit et al in [4] to study fuzzy groups, respectively. In the present paper, we use the notion of level subsets to study fuzzy semigroups, which were introduced by Kuroki in [5] first. For convenient, we show some basic definitions and results in the following.

Definition 1.1. Let S be a set. A fuzzy set of S is a function $A : S \longrightarrow [0, 1]$.

Definition 1.2. Let A be a fuzzy set of a set S. For $t \in [0, 1]$, the set

$$A_t = \{x \in S \mid A(x) \geqslant t\}$$

is called a level subset of A.

Proposition 1.3. Let A be a fuzzy set of a set S. Then two level subsets A_{t_1} and A_{t_2} (with $t_1 < t_2$) of A are equal if and only if there is no $x \in S$ such that $t_1 \leq A(x) < t_2$.

Proof. Let $A_{t_1} = A_{t_2}$. If $A_{t_1} = A_{t_2} = \phi$, then there is no $x \in S$ such that $t_1 \leq A(x) < t_2$ obviously. If $A_{t_1} = A_{t_2} \neq \phi$, suppose there exists $x \in S$ such that $t_1 \leq A(x) < t_2$, then $x \in A_{t_1}$, but $x \notin A_{t_2}$. This is in contradiction with $A_{t_1} = A_{t_2}$. Conversely. By $A_{t_1} \supseteq A_{t_2}$ since $t_1 < t_2$, if $A_{t_1} = \phi$, then $A_{t_2} = \phi = A_{t_1}$. If $A_{t_1} \neq \phi$, then we must have $A_{t_1} = A_{t_2}$. Otherwise, there exists $x \in A_{t_1}$, but $x \notin A_{t_2}$, then $t_1 \leq A(x) < t_2$. That is, there is $x \in S$ such that $t_1 \leq A(x) < t_2$, which contradicts the hypothesis.

Proposition 1.4. Let A be a fuzzy set of a set S. If $\mathrm{Im}(A) = \{t_1, t_2, \cdots, t_n\}$ with $t_1 > t_2 > \cdots > t_n$. Then $\{A_{t_1}, A_{t_2}, \cdots, A_{t_n}\}$ contains all nonempty level subsets of A.

Proof. Let A_t be a nonempty level subset of A. Then by $\mathrm{Im}(A) = \{t_1, t_2, \cdots, t_n\}$ we can let

$$t_j = \min\{A(x) \mid A(x) \geq t, x \in S\} \qquad (1)$$

for some j, $1 \leq j \leq n$. Therefore there is $x \in A_t$ such that $A(x) = t_j$, so $A_{t_j} \subseteq A_t$ since $t_j \geq t$. Now let $y \in A_t$, thus $A(y) \geq t$. Let $A(y) = t_i$ for some i, $1 \leq i \leq n$. Then $y \in A_{t_i}$. By (1) it must have $t_i \geq t_j$, thus $y \in A_{t_i} \subseteq A_{t_j}$. That is, $y \in A_{t_j}$. Hence $A_t \subseteq A_{t_j}$. Thereby $A_t = A_{t_j}$ for some j, $1 \leq j \leq n$.

Definition 1.5. A fuzzy set A of a semigroup S is called a fuzzy semigroup of S if

$$A(xy) \geq \min\{A(x), A(y)\}$$

for all x, $y \in S$.

2. level subsemigroups

The following proposition is obvious (cf. [6]), its proof is omitted.

Proposition 2.1. A fuzzy set A of a semigroup S is a fuzzy semigroup if and only if every nonempty level subset A_t, $t \in [0, 1]$, of A is a subsemigroup of S.

Definition 2.2. Let A be a fuzzy set of a semigroup S. The subsemigroups A_t, $t \in [0, 1]$, are called level subsemigroups of A.

By Proposition 2.1, the proof of the following two propositions are immediate from Proposition 1.3, 1.4, respectively.

Proposition 2.3. Let A be a fuzzy semigroup of a semigroup S. Then two level subsemigroups A_{t_1} and A_{t_2} (with $t_1 < t_2$) of A are equal if and only if there is no $x \in S$ such that $t_1 \leq A(x) < t_2$.

Proposition 2.4. Let A be a fuzzy subsemigroup of a semigroup S. If Im(A) = $\{t_1, t_2, \cdots, t_n\}$ with $t_1 > t_2 > \cdots > t_n$, then $\{A_{t_1}, A_{t_2}, \cdots, A_{t_n}\}$ contains all level subsemigroups of A.

Proposition 2.5. Any subsemigroup H of a semigroup S can be realised as a level subsemigroup of some fuzzy semigroup of S.

Proof. Let A be a fuzzy set of S defined by

$$A(x) = \begin{cases} t, & \text{if } x \in H \\ 0, & \text{otherwise} \end{cases}$$

for all $x \in S$. Then $A_t = H$. In the following, we shall prove that A is a fuzzy semigroup of S. Let $x, y \in S$, if $x, y \in H$, then $xy \in H$ since H is a subsemigroup of S. So

$$A(xy) = t = \min\{A(x), A(y)\}.$$

If $x \notin H$ or $y \notin H$, then $A(x) = 0$ or $A(y) = 0$. It is clearly that

$$A(xy) \geq 0 = \min\{A(x), A(y)\}.$$

So far we have proved

$$A(xy) \geqslant \min\{A(x),\ A(y)\}$$

for all $x, y \in S$. Thereby A is a fuzzy semigroup of S.

Proposition 2.6. Let $\bar{A}$ be the collection of all fuzzy semigroups of a semigroup S and $\bar{B}$ be the collection of all level subsemigroups of members of $\bar{A}$. Then there is a one-to-one correspondence between the subsemigroups of S and the equivalence classes of level subsemigroups (under a suitable equivalence relation on $\bar{B}$).

Proof. Similar to Theorem 3.3 of [3].

3. The chain of level subsemigroups

Definition 3.1. A nonempty subset T of a semigroup S is called a proper subsemigroup of S if $a \in T$ and $b \in T$ imply $ab \in T$, and $T \subset S$ (that is, $T \subseteq S$ but $T \neq S$).

Proposition 3.2. Let A be a fuzzy semigroup of a semigroup S. If $Im(A) = \{t_1, t_2, \cdots, t_n\}$ with $t_1 > t_2 > \cdots > t_n$, then there exists a unique finite chain of level subsemigroups of A:

$$A_{t_1} \subset A_{t_2} \subset \cdots \subset A_{t_n} = S,$$

where every A_{t_i}, $i = 1, 2, \cdots, n-1$, is a proper subsemigroup of S.

Proof. Since $Im(A) = \{t_1, t_2, \cdots, t_n\}$ with $t_1 > t_2 > \cdots > t_n$, we have that every A_{t_i}, $i = 1, 2, \cdots, n$, is a level subsemigroup of A by Proposition 2.1, and $A_{t_1} \subseteq A_{t_2} \subseteq \cdots \subseteq A_{t_n} = S$. If $1 \leq j < i \leq n$, then there must exist $x \in S$ such that $A(x) = t_i$ since $t_i \in Im(A)$. That is, $t_i = A(x) < t_j$. Hence $A_{t_i} \neq A_{t_j}$ by Proposition 2.3. Thereby $A_{t_j} \subset A_{t_i}$. Therefore we have

$$A_{t_1} \subset A_{t_2} \subset \cdots \subset A_{t_n} = S.$$

By Proposition 2.4, A_{t_i}, $i = 1, 2, \cdots, n$, are all level subsemigroups of A.

Hence

$$A_{t_1} \subset A_{t_2} \subset \cdots \subset A_{t_n} = S$$

is the only chain of properly inclusive level subsemigroups of A.

Proposition 3.3. Let A be a fuzzy semigroup of a semigroup S. If $Im(A) = \{t_1, t_2, \cdots, t_n, \cdots\}$ with $t_1 > t_2 > \cdots > t_n > \cdots \geqslant 0$, then there must exist a chain of level subsemigroups of A:

$$A_{t_1} \subset A_{t_2} \subset \cdots \subset A_{t_n} \subset \cdots \subset S,$$

where every A_{t_i}, $i = 1, 2, \cdots$, is a proper subsemigroup of S.

Proof. It is immediate from Proposition 2.1 and 2.3.

Proposition 3.4. Let S be a finite semigroup and $f: S \longrightarrow T$ be an onto homomorphism . Let A be a fuzzy semigroup of S with $Im(A) = \{t_0, t_1, \cdots, t_n\}$ and $t_0 > t_1 > \cdots > t_n$. If the chain of level subsemigroups of A is

$$A_{t_0} \subseteq A_{t_1} \subseteq \cdots \subseteq A_{t_n} = S,$$

then the chain of level subsemigroups of B, the homomorphic image of A under f, will be

$$f(A_{t_0}) \subseteq f(A_{t_1}) \subseteq \cdots \subseteq f(A_{t_n}) = T.$$

The proof of above Proposition 3.4 is similar to the proof of Theorem 4.6 in [4].

Proposition 3.5. If one of the following for a given chain of proper subsemigroups of a semigroup S holds:

(1) $S_0 \subset S_1 \subset \cdots \subset S_n = S$ and $1 \geqslant t_0 > t_1 > \cdots > t_n \geqslant 0$;

(2) $S = S_0 \supset S_1 \supset \cdots \supset S_n$ and $0 \leqslant t_0 < t_1 < \cdots < t_n \leqslant 1$;

(3) $S_0 \subset S_1 \subset \cdots \subset S_n \subset \cdots \subset S$ and $1 \geqslant t_0 > t_1 > \cdots > t_n > \cdots \geqslant 0$;

(4) $S = S_0 \supset S_1 \supset \cdots \supset S_n \supset \cdots \supset S_\infty$ and $0 \leqslant t_0 < t_1 < \cdots < t_n < \cdots < t_\infty \leqslant 1$.

Then there is a fuzzy semigroup A of S such that $A_{t_i} = S_i$ for all i.

Proof. For (1), (2), (3), (4), we respectively define a fuzzy set A of S as followings:

(1)
$$A(x) = \begin{cases} t_0, & \text{if } x \in S_0 \\ t_i, & \text{if } x \in S_i \setminus S_{i-1},\ i = 1, 2, \cdots, n \end{cases}$$

(2)
$$A(x) = \begin{cases} t_i, & \text{if } x \in S_i \setminus S_{i+1},\ i = 0, 1, \cdots, n-1 \\ t_n, & \text{if } x \in S_n \end{cases}$$

(3)
$$A(x) = \begin{cases} t_0, & \text{if } x \in S_0 \\ t_i, & \text{if } x \in S_i \setminus S_{i-1},\ i = 1, 2, \cdots \\ 0, & \text{if } x \in S \setminus \bigcup_{i=1}^{\infty} S_i \end{cases}$$

(4)
$$A(x) = \begin{cases} t_i, & \text{if } x \in S_i \setminus S_{i+1},\ i = 0, 1, 2, \cdots \\ t_\infty, & \text{if } x \in S_\infty \end{cases}$$

for all $x \in S$. We only prove (1), the proof of other cases are similar. First we show that $A_{t_i} = S_i$ for all i = 0, 1, ⋯, n. Obviously, $A_{t_0} = S_0$. For $i \geq 1$, we have

$$\begin{aligned} A_{t_i} &= \{x \in S \mid A(x) \geq t_i\} \\ &= \bigcup_{j=0}^{i} \{x \in S \mid A(x) = t_j\} \\ &= S_0 \cup (S_1 \setminus S_0) \cup \cdots \cup (S_i \setminus S_{i-1}) \\ &= S_i. \end{aligned}$$

Let $x, y \in S$. If $A(x) = t_n$ or $A(y) = t_n$, then

$$A(xy) \geq t_n = \min\{A(x), A(y)\}.$$

If $A(x) \neq t_n$, and $A(y) \neq t_n$ also. Let $A(x) = t_i$, $A(y) = t_j$, and $t_n < t_i \leq t_j$. Then $y \in A_{t_j} \subseteq A_{t_i}$ since $t_i \leq t_j$, and $x \in A_{t_i}$. Thus $xy \in A_{t_i}$ since S_i is a subsemigroup of S. Hence

$$A(xy) \geqslant t_i = \min\{t_i, t_j\} = \min\{A(x), A(y)\}.$$

Therefore

$$A(xy) \geqslant \min\{A(x), A(y)\}$$

for all $x, y \in S$. That is, A is a fuzzy semigroup of S. This completes the proof.

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