Direct product of HX groups and HX groups on direct product group

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Abstract

In the paper [1] the upgrade of the structure of groups has been considered, in which the concept of HX group has been raised. In this paper we will discuss the direct product of HX groups and HX gruops on direct product group.

Key words: HX group, structure of group, direct product.

§1. Prepare knowledge

The paper [1] gived the concept of HX group. The so-called HX group is a group which is formed on the power set P(G) of a group G.

Definition 1.1. Let $(G,\cdot)$ be a group. The binary operation of the group G:

$$G\times G \to G\ ,\ (a,b)\to a\cdot b=ab$$

can be naturally induced to $P(G)$: for any $A,B\in P(G)$,

$$AB=\{ab\mid a\in A,b\in B\} \tag{1.1}$$

Let $\mathcal{G}\in P_0(G)=P(G)-\{\phi\}$, $\mathcal{G}\neq\phi$. $\mathcal{G}$ is called a HX group on G iff it forms a group for the operation (1.1), which its unit element is denoted by E. Let $A\in\mathcal{G}$, the inverse element of A is denoted by A^{-1}.

Obviously, E is a sub-semigroup of G.

Definition 1.2. Let $\mathcal{G}$ be a HX group on G. $\mathcal{G}$ is called a regular HX group iff its unit elemente E is a monoid of G.

Theorem 1.1. Let $\mathcal{G}$ be a HX group on G. Then

1) $(\forall A\in\mathcal{G})\ (|A|=|E|)$

2) $(\forall A,B\in\mathcal{G})\ (A\cap B\neq\phi\Rightarrow|A\cap B|=|E|)$

The paper [2] gived the concept of uniform HX group. For $A\in P_0(G)$,

$$A^{\ominus}=\{x^{-1}\mid x\in A\}$$

is called inverse set of A.

Definition 1.3. A HX group $\mathcal{G}$ on G is called the uniform HX group iff for any $A\in\mathcal{G}$, $A^{\ominus}=A^{-1}$.

Theorem 1.2. Let $\mathcal{G}$ be a HX group on G, $\mathcal{G}$ is uniform iff the unit element E of $\mathcal{G}$ is a subgroup of G.

Theorem 1.3. Let $\mathfrak{G}$ be an uniform HX group on G. $G^*=\cup\{A \mid A\in\mathfrak{G}\}$. Then

1) G^* is a subgroup of G.

2) E is a normal subgroup of G^*.

3) $\mathfrak{G}=G^*/E$.

§2. Direct product of HX groups and HX groups on direct product group

First of all, we will deal with the case of external direct product. Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be respectively HX groups on G_1 and G_2. The direct product of $\mathfrak{G}_1$ and $\mathfrak{G}_2$ is

$$\mathfrak{G}_1\times\mathfrak{G}_2=\{(A_1,A_2)\mid A_1\in\mathfrak{G}_1, A_2\in\mathfrak{G}_2\}.$$

Let $\mathfrak{G}$ be a HX group on the direct product group $G_1\times G_2$. In this paper we will discuss the relation between $\mathfrak{G}_1\times\mathfrak{G}_2$ and $\mathfrak{G}$.

Theorem 2.1. Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be respectively HX groups on G_1 and G_2. Then

$$\cup\{A_1\times A_2\mid (A_1,A_2)\in\mathfrak{G}_1\times\mathfrak{G}_2\}$$

is a sub-semigroup of $G_1\times G_2$.

Proof. Suppose $(a_1,a_2),(b_1,b_2)\in\cup\{A_1\times A_2\mid(A_1,A_2)\in\mathfrak{G}_1\times\mathfrak{G}_2\}$. Then $(a_1,a_2)\in A_1\times A_2$, $(b_1,b_2)\in B_1\times B_2$, where $(A_1,A_2),(B_1,B_2)\in\mathfrak{G}_1\times\mathfrak{G}_2$. We have $a_1\in A_1, a_2\in A_2, b_1\in B_1, b_2\in B_2$. Therefore,

$$(a_1,a_2)(b_1,b_2)=(a_1b_1,a_2b_2)\in A_1B_1\times A_2B_2,$$

where $(A_1B_1,A_2B_2)\in\mathfrak{G}_1\times\mathfrak{G}_2$. So, it is proved that

$$\cup\{A_1\times A_2 \mid (A_1, A_2)\in \mathfrak{G}_1\times\mathfrak{G}_2\}$$

is a sub-semigroup of $G_1\times G_2$.

Theorem 2.2. Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be respectively regular(uniform) HX groups on G_1 and G_2 . Then

$$\cup\{A_1\times A_2 \mid (A_1, A_2)\in \mathfrak{G}_1\times\mathfrak{G}_2\}$$

is a monoid(subgroup) of $G_1\times G_2$.

Theorem 2.3. Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be respectively HX groups on G_1 and G_2. Then

$$\mathfrak{G}=\{A_1\times A_2 \mid (A_1, A_2)\in \mathfrak{G}_1\times\mathfrak{G}_2\}$$

is a HX group on $G_1\times G_2$.

Proof. Suppose $A_1\times A_2, B_1\times B_2\in\mathfrak{G}$, we can prove that

$$(A_1\times A_2)(B_1\times B_2)=(A_1B_1)\times(A_2B_2).$$

In fact, because $(a_1,a_2)(b_1,b_2)=(a_1b_1,a_2b_2)$,

$(a_1,a_2)(b_1,b_2)\in(A_1\times A_2)(B_1\times B_2)$ iff $(a_1b_1,a_2b_2)\in(A_1B_1)\times(A_2B_2)$.

To take note of $(A_1B_1,A_2B_2)\in\mathfrak{G}_1\times\mathfrak{G}_2$. Therefore,

$$(A_1\times A_2)(B_1\times B_2)=(A_1B_1)\times(A_2B_2)\in\mathfrak{G}.$$

Thus $\mathfrak{G}$ is close for its operation.

Let (E_1,E_2) be the unit element of $\mathfrak{G}_1\times\mathfrak{G}_2$, where E_1 and E_2 are respectively the unit elements of $\mathfrak{G}_1$ and $\mathfrak{G}_2$. Obviously, $E_1\times E_2$ is the unit element of $\mathfrak{G}$.

The inverse of $A_1\times A_2$ is $A_1^{-1}\times A_2^{-1}$.

So, $\mathfrak{G}$ is a HX group on $G_1\times G_2$.

Theorem 2.4. Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be respectively regular(uniform) HX groups on G_1 and G_2. Then

$$\mathcal{G} = \{A_1 \times A_2 \mid (A_1, A_2) \in \mathcal{G}_1 \times \mathcal{G}_2\}$$

is a regular(uniform) HX group on $G_1 \times G_2$.

Let $\mathcal{G}$ be a HX group on $G_1 \times G_2$. For $\mathbf{A} \in \mathcal{G}$,

$$\mathbf{A}_{G_1} = \{a_1 \mid \exists (a_1, a_2) \in \mathbf{A}\}$$

is called the projection of $\mathbf{A}$ in G .

$$\mathcal{G}_{G_1} = \{\mathbf{A}_{G_1} \mid \mathbf{A} \in \mathcal{G}\}$$

is called the projection of $\mathcal{G}$ in P(G).

$$\mathbf{A}_{G_2} = \{a_2 \mid \exists (a_1, a_2) \in \mathbf{A}\}$$

is called the projection of $\mathbf{A}$ in G .

$$\mathcal{G}_{G_2} = \{\mathbf{A}_{G_2} \mid \mathbf{A} \in \mathcal{G}\}$$

is called the projection of $\mathcal{G}$ in P(G).

Theorem 2.5. Let $\mathcal{G}$ be a HX group on $G_1 \times G_2$. Then

1) $\mathcal{G}_{G_1}$ is a HX group on G_1.

2) $\mathcal{G}_{G_2}$ is a HX group on G_2.

Proof. We prove only 1). For any $\mathbf{A}_{G_1}, \mathbf{B}_{G_1} \in \mathcal{G}_{G_1}$, where $\mathbf{A}, \mathbf{B} \in \mathcal{G}$, we can prove that $\mathbf{A}_{G_1} \mathbf{B}_{G_1} = (\mathbf{AB})_{G_1}$, where $\mathbf{AB} \in \mathcal{G}$. That is $\mathcal{G}_{G_1}$ is close for its operation.

It is obvious that $\mathbf{E}_{G_1}$ is the unit element of $\mathcal{G}_{G_1}$, where $\mathbf{E}$ is the unit element of $\mathcal{G}$.

For any $\mathbf{A}_{G_1} \in \mathcal{G}_{G_1}$, the inverse element of $\mathbf{A}_{G_1}$ is $\mathbf{A}^{-1}_{G_1}$, i.e., $(\mathbf{A}_{G_1})^{-1} = \mathbf{A}^{-1}_{G_1}$.

Therefore, $\mathcal{G}_{G_1}$ is a HX group on G_1.

Theorem 2.6. Let $\mathcal{G}$ be a regular(uniform) HX group on $G_1 \times G_2$. Then

1). $\mathcal{G}_{G_1}$ is a regular(uniform) HX group on G_1.

2). $\mathcal{G}_{G_2}$ is a regular(uniform) HX group on G_2.

Let $\mathcal{G}$ be a HX group on $G_1 \times G_2$. By theorem 2.5, we obtain that

$$\mathcal{G}_{G_1} = \{A_{G_1} \mid A \in \mathcal{G}\}, \quad \mathcal{G}_{G_2} = \{A_{G_2} \mid A \in \mathcal{G}\}$$

are respectively Hx groups on G_1 and G_2. By theorem 2.3, we obtain a HX group on $G_1 \times G_2$:

$$\mathcal{G}' = \{A_{G_1} \times A_{G_2} \mid A \in \mathcal{G}\}$$

from $\mathcal{G}_{G_1} \times \mathcal{G}_{G_2}$.

We point out that usually $\mathcal{G}' \neq \mathcal{G}$.

Example. Let $G_1 = G_2 = (a)$, where a is a second order element. We let $E = \{(e,e),(a,a)\}$, $A = \{(e,a),(a,e)\}$. It is obvious that $\mathcal{G} = \{E, A\}$ is a HX group on $G_1 \times G_2$. Because $\mathcal{G}_{G_1} = \{E_{G_1}, A_{G_1}\} = \{\{e,a\}\}$ and $\mathcal{G}_{G_2} = \{E_{G_2}, A_{G_2}\} = \{\{e,a\}\}$, $\mathcal{G}' = \{\{(e,e),(e,a),(a,e),(a,a)\}\} = \{G_1 \times G_2\}$. But $\mathcal{G}' \neq \mathcal{G}$.

Gerenally, we have that $\mathcal{G}' = \mathcal{G}$ iff for any $A \in \mathcal{G}$, $A = A_{G_1} \times A_{G_2}$.

Theorem 2.7. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be respectively HX groups on G_1 and G_2,

$$\mathcal{G} = \{A_1 \times A_2 \mid (A_1, A_2) \in \mathcal{G}_1 \times \mathcal{G}_2\}.$$

Then $\mathcal{G}_{G_1} = \mathcal{G}_1$, $\mathcal{G}_{G_2} = \mathcal{G}_2$.

Lastly, we will discuss the case of internal direct product. Let a group G be the internal direct product group of its subgroups G_1 and G_2. That is

1). $G = G_1 G_2$.

2). $G_1 \cap G_2 = \{e\}$

3). For any $a_1 \in G_1$, $a_2 \in G_2$: $a_1 a_2 = a_2 a_1$.

Let $A \subsetneq G$, A is called be close for direct product decomposition of G, if

$$A=(A\cap G_1)(A\cap G_2)$$

holds.

Theorem 2.8. Let G be the internal direct product group of its subgroups G_1 and G_2 . $\mathcal{G}$ is a HX group on G and for any $A\in\mathcal{G}$,

$$A\cap G_1\neq\phi,\quad A\cap G_2\neq\phi.$$

Then

$$\mathcal{G}_1=\{A\cap G_1\mid A\in\mathcal{G}\},\quad \mathcal{G}_2=\{A\cap G_2\mid A\in\mathcal{G}\}$$

are respectively HX groups on G_1 and G_2 . If any element of $\mathcal{G}$ is close, then $\mathcal{G}=\mathcal{G}_1\mathcal{G}_2$.

Proof. It is proved obviously that for any $A,B\in\mathcal{G}$, we have

$$(A\cap G_1)(B\cap G_1)=(AB)\cap G_1 .$$

Let E be the unit element of $\mathcal{G}$. Then $E\cap G_1$ is the unit element of $\mathcal{G}_1$. $A^{-1}\cap G_1$ is the inverse element of $A\cap G_1$. So $\mathcal{G}_1$ is a HX group on G_1. Similarly, we can prove that $\mathcal{G}_2$ is a HX group on G_2. Because any element of $\mathcal{G}$ is close, i.e., for any $A\in\mathcal{G}$, $A=(A\cap G_1)(A\cap G_2)$, $\mathcal{G}=\mathcal{G}_1\mathcal{G}_2$.

The proof is finished.

Since G is the internal direct product of its subgroups G_1 and G_2, any element of G is represented for production of elements of G_1 and G_2, and represention is only. Let $A\in P(G)$, the projections of A in G_1 and G_2 are respectively

$$A_{G_1}=\{a_1\mid\exists a_2\in G_2, a_1a_2\in A\},\quad A_{G_2}=\{a_2\mid\exists a_1\in G_1, a_1a_2\in A\}.$$

Let $\mathcal{G}$ be a HX group on G, the projections of $\mathcal{G}$ in $P(G_1)$ and $P(G_2)$ are respectively

$$\mathcal{G}_{G_1}=\{A_{G_1} \mid A\in\mathcal{G}\},\quad \mathcal{G}_{G_2}=\{A_{G_2} \mid A\in\mathcal{G}\}.$$

Theorem 2.9. Let G be the internal direct product of its subgroups G_1 and G_2, and let $\mathcal{G}$ be a HX group on G. Then $\mathcal{G}_{G_1}$ and $\mathcal{G}_{G_2}$ are respectively HX groups on G_1 and G_2, and $\mathcal{G}=\mathcal{G}_{G_1}\mathcal{G}_{G_2}$ holds.

Proof. It is obviously proved that for any $A,B\in\mathcal{G}$ we have

$$A_{G_1}B_{G_1}=(AB)_{G_1},\quad A_{G_2}B_{G_2}=(AB)_{G_2},\quad A=A_{G_1}A_{G_2}.$$

Thus we can finish the proof.

In theorem 2.8 and theorem 2.9, for the HX group $\mathcal{G}$ on G we obtained respectively HX groups $\mathcal{G}_1$ and $\mathcal{G}_2$ as well as $\mathcal{G}_{G_1}$ and $\mathcal{G}_{G_2}$ on G_1 and G_2, and $\mathcal{G}=\mathcal{G}_1\mathcal{G}_2$, $\mathcal{G}=\mathcal{G}_{G_1}\mathcal{G}_{G_2}$. It is well worth noticing that $\mathcal{G}$ is not the internal direct product of $\mathcal{G}_1$ and $\mathcal{G}_2$ or $\mathcal{G}_{G_1}$ and $\mathcal{G}_{G_2}$.

Theorem 2.10. Let G be the internal direct product of its subgroups G_1 and G_2, and let $\mathcal{G}$ be a HX group on G. Then the necessary conditions that $\mathcal{G}$ is a internal direct product of a HX group $\mathcal{G}_1$ on G_1 and a HX group $\mathcal{G}_2$ on G_2 are that any element of $\mathcal{G}$ is a set of simple point and $\mathcal{G}$ is uniform.

Proof. Let $\mathcal{G}$ be the internal direct product of its subgroups $\mathcal{G}_1$ and $\mathcal{G}_2$, and let $\mathcal{G}_1$ and $\mathcal{G}_2$ be respectively HX groups on G_1 and G_2. Then

$$\mathcal{G}_1\cap\mathcal{G}_2=\{E\}$$

where E is the unit element of $\mathcal{G}$. We thus have

$$E\subseteq G_1\cap G_2=\{e\}$$

Since $E\neq\emptyset$, $E=\{e\}$. E is a subgroup of G. From theorem 1.2, $\mathcal{G}$ is uniform. From theorem 1.1, for any $A\in\mathcal{G}$, $|A|=|E|$ holds. Therefore, any element of $\mathcal{G}$ is a set of simple point.

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