WEAK FUZZY TOPOLOGICAL SPACES

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1. INTRODUCTION

In the area of fuzzy topology, much famous research has bee carried. Prof. Liu Ying-ming has made a greater contribution to fuzzy topology in [2],[3] and other famous papers. It should be noticed that $\mathcal{J}$ is a classical subset of $\mathcal{F}(X)$(a set of all fuzzy subset over X) in a fuzzy topological space $(X,\mathcal{J})$ and $\mathcal{J}$ is a special lattice topology. So it seems to be necessary that we should introduce a new kind of fuzzy topological spaces $(X,\underline{J})$, where $\underline{J}$ is a fuzzy subset of $\mathcal{P}(X)$(power set of X) and $\underline{J}$ has similiar topological structures to classical topology and fuzzy topology. In this paper, we give a concept of weak fuzzy topology. In weak fuzzy topological space $(X,\underline{J})$, $\underline{J}$ is a fuzzy subset of $\mathcal{P}(X)$ and each λ-strong cut set $\underline{J}_{\dot{\lambda}}$ is a classical topology over X. By the use of weak fuzzy topology, we introduce some concepts such as interior set, closure, open set, closed set, neighborhood and continuity. Clearly, many concepts and conclusions in classical topology can easily be generalized to weak fuzzy topological spaces.

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2. WEAK FUZZY TOPOLOGICAL SPACE

Definition 1. A mapping $\underline{J}: \mathcal{P}(X) \longrightarrow [0, 1]$ is called as a weak fuzzy topology(w.f.t.) over X if strong cut set $\underline{J}_{\dot\lambda}=\{A|A\in\mathcal{P}(X) \text{ and } \underline{J}(A)>\lambda\}$ is a topology over X, for any $\lambda\in[0, 1)$, then $(X, \underline{J})$ is called a weak fuzzy topological space(w.f.t.s.).

Proposition 1. Let $(X, \underline{J})$ be a w.f.t.s., then

(1) $\underline{J}_{\lambda}=\{A|A\in\mathcal{P}(X) \text{ and } \underline{J}(A)\geq\lambda\}$ is a topology over X, for any $\lambda\in[0, 1]$;

(2) (i) $\underline{J}(X)=\underline{J}(\phi)=1$; (ii) $\underline{J}(A\cap B)\geq \mathrm{Min}\{\underline{J}(A),\underline{J}(B)\}$, for any $A,B\in\mathcal{P}(X)$; (iii) $\underline{J}(\bigcup_{t\in T} A_t)\geq\inf_{t\in T}\underline{J}(A_t)$, for any $A_t\in\mathcal{P}(X)$.

Definition 2. Let $\underline{J}$ be a w.f.t. over X and $\underline{B}\subseteq\underline{J}$ be a fuzzy subset of $\mathcal{P}(X)$. $\underline{B}$ is called as a subbase of $\underline{J}$ if $\underline{B}_{\dot\lambda}$ is a subbase of $\underline{J}_{\dot\lambda}$, for any $\lambda\in[0, 1)$.

Proposition 2. Let $\underline{B}$ be a subbase of w.f.t. $\underline{J}$ and $A\in\mathcal{P}(X)$. Then we have: if $\underline{J}(A)>\lambda$, then there exists some sets $A_t\in\mathcal{P}(X)$, satisfying $\underline{B}(A_t)>\lambda$ and $A=\bigcup_{t\in T} A_t$.

Definition 3. Let $\underline{J}$ be a w.f.t. over X and $A\in\mathcal{F}(X)$, $H_A(\lambda)=\bigcup\{U\in\mathcal{P}(X)|U\subseteq A_{\dot\lambda} \text{ and } U\in\underline{J}_{\dot\lambda}\}$, for any $\lambda\in[0, 1)$, then $A^{\circ}=\bigcup_{\lambda\in[0,1]}\lambda H_A(\lambda)$ is called as interior set of A, $\bar{A}=((A^c)^{\circ})^c$ is called as closure of A.

Definition 4. Let $\underline{J}$ be a w.f.t. over X and $A,B\in\mathcal{F}(X)$, then

(1) A is called as a $\underline{J}$-open set if $A_{\dot\lambda}\in\underline{J}_{\dot\lambda}$, for any $\lambda\in[0, 1)$.

(2) A is called as a $\underline{J}$-closed set if A^c is a $\underline{J}$-open set.

Theorem 1. (1) A° is a $\underline{J}$-open set; (2) $A^{\circ}\subseteq A$; (3) if U is a $\underline{J}$-open set and $U\subseteq A$, then $U\subseteq A^{\circ}$; (4) $A^{\circ}=\bigcup\{U\in\mathcal{F}(X)|U\subseteq A$ and U is weak fuzzy topology(w.f.t.) over X if strong cut set $\underset{\cdot}{\underline{J}}_\lambda=\{A\mid A\in\mathcal{P}(X) \text{ and } \underline{J}(A)>\lambda\}$ is a topology over X, for any $\lambda\in[0,1)$, then $(X,\underline{J})$ is called a weak fuzzy topological space(w.f.t.s.).

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Theorem 2. (1) $\bar{A}$ is a $\underline{J}$-closed set; (2) $\bar{A} \supseteq A$; (3) if B is a $\underline{J}$-closed set and $B \supseteq A$, then $B \supseteq \bar{A}$; (4) $\bar{A} = \bigcap \{B \in \mathscr{F}(X) \mid B \supseteq A$ and B is a $\underline{J}$-closed set$\}$.

Theorem 3. (1) $(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}$; (2) $(\overline{A \cup B}) = \bar{A} \cup \bar{B}$.

Definition 5. (1) if $A(x)=1$ or $x \in A_{\lambda}$, then we say fuzzy point x_{λ} belongs to A. It's denoted as $x_{\lambda} \in A$.

(2) if $U \subseteq A$, $U_{\lambda} \in \underline{J}_{\lambda}$ and $x_{\lambda} \in U$, then U is called as neighborhood of fuzzy point x_{λ}.

Theorem 4. A is a neighborhood of x_{λ} if and only if there exists a $\underline{J}$-open set $U \subseteq A$ and $x_{\lambda} \in U$.

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Definition 6. Let $\underline{X}$ be a set of all fuzzy points over X and mapping $N: \underline{X} \longrightarrow \mathscr{P}(\mathscr{F}(X))$ $x_{\lambda} \longrightarrow N(x_{\lambda})$ satisfies:

(1) if $A, B \in N(x_{\lambda})$, then $A \cap B \in N(x_{\lambda})$;

(2) if $A \in N(x_{\lambda})$ and $A \subseteq B$, then $B \in N(x_{\lambda})$;

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Let $J = \{A \mid A \in N(y_{\mu})$, for any $y_{\mu} \in A\} \cup \{\phi\}$, we have:

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3. CONTINUITY

Definition 7. Let $(X,\underline{J})$ and $(X',\underline{J}')$ be two w.f.t.s.. $f:X\longrightarrow X'$ is called as continuous mapping if f is a continuous mapping from $(X,\underline{J}_{\lambda})$ to $(X',\underline{J}'_{\lambda})$, for any $\lambda\in[0,1)$.

Theorem 6. The following propositions are equivalent.

(1) $f:(X,\underline{J})\longrightarrow(X',\underline{J}')$ is a continuous mapping;

(2) if A is a $\underline{J}'$-open set, then $f^{-1}(A)$ is a $\underline{J}$-open set;

(3) if B is a $\underline{J}'$-closed set, then $f^{-1}(B)$ is a $\underline{J}$-closed set

(4) for any $W\in N(f(x_{\lambda}))$, there exists $V\in N(x_{\lambda})$ such that $f(U)\subseteq W$.

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(2) $f(\bar{A})\subseteq\overline{f(A)}$, for any $A\in\mathcal{F}(X)$;

(3) $\overline{f^{-1}(B)}\subseteq f^{-1}(\bar{B})$, for any $B\in\mathcal{F}(X')$;

(4) $f^{-1}(B^{\circ})\subseteq(f^{-1}(B))^{\circ}$, for any $B\in\mathcal{F}(X')$.

Clearly, many concepts and conclusions of classical topology can easily be generalized to w.f.t.s.. In the following papers, we shall discuss them.

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[1] C. L. Chang, Fuzzy Topological Spaces, Journal of Mathematical Analysis and Applications, 24, 182-190(1968).

[2] Pu Pao-Ming & Liu Ying-Ming, Fuzzy Topology I. Neighborhood Structure of a Fuzzy Point and Moore-Smith Convergence, Journal of Mathematical Analysis and

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