THE RESEARCH OF SPECIAL FUZZY MATRICES

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ABSTRACT

This paper introduces several kinds of special fuzzy matrices and gives a few decision theorems which are used to discriminate these matrices.

Keywords : Special fuzzy matrix

1 PREPARATORY KNOWLEDGES

Let's now discuss the problems on a semisimple ring $S=([0,1], +, \cdot)$, where $a+b=\max\{a,b\}$, $a\cdot b=\min\{a,b\}$ for $\forall a,b\in[0,1]$.

Definition 1.1 For any m×n fuzzy matrix A, the set $R(A)$ consisting of all linear combinations of all row vectors of A is called row space of A. Row rank $\rho r(A)$ of A is the number of vectors of minimum generating basis of $R(A)$. Column rank $\rho c(A)$ and column space $C(A)$ may be defined similarly. Let A be a n×n fuzzy matrix, if $\rho r(A)=\rho c(A)=t$, then the number t is called rank of A and expressed as $\rho(A)$.

For the convenience of statement, we define that $\mathcal{M}_{m\times n}$ expresses a set consisting of all m×n fuzzy matrices.

Definition 1.2 For $\forall A\in\mathcal{M}_{m\times n}$, the schein rank $\rho s(A)$ of A is the minimum number of the fuzzy matrices which have rank one and the sum of which is equal to A just.

Theorem 1.1 [2] Let $A\in\mathcal{M}_{m\times n}$ and A be nonzero matrix, then $\rho s(A)=s$ iff the fuzzy relational non-deterministic equation of A

$$A= X_{m\times t}\cdot Y_{t\times n} \qquad (1.1)$$

(1) has solution when exponent t=s.

(2) has no solution when exponent t satisfies inequality

$$1\leq t\leq s-1$$

In the equation (1.1), the A is known fuzzy matrix but X and Y are all unknown ones. And the subscript t is called exponent of non-deterministic equation of A.

The fuzzy matrices $X_{m\times s}$ and $Y_{s\times n}$ satisfying (1.1) is called solution matrix of non-deterministic equation of A when exponent t is s.

2 FULL-RANK FUZZY MATRIX

Definition 2.1 Let $A=(a_{ij})_{m\times n}\in\mathcal{M}_{m\times n}$, A is called row full-rank if $\rho r(A)=m$; and A is called column full-rank if $\rho c(A)=n$.

Definition 2.2 Let $A\in\mathcal{M}_{n\times n}$, then A is called full-rank if $\rho r(A)=\rho c(A)=\rho(A)=n$.

From the definitions 1.1, 1.2 and 2.2, the following theorem 2.1 hold true.

Theorem 2.1 Let $A\in\mathcal{M}_{n\times n}$, then A is full-rank iff $\rho(A)=\rho s(A)=n$.

Theorem 2.2 Let $A\in\mathcal{M}_{n\times n}$, then A is full-rank iff fuzzy relational nondeterministic equation of A.

$$A=X_{n\times t}\cdot Y_{t\times n} \qquad (2.1)$$

has no solution when t=n-1.

Proof Sufficiency. Suppose that the equation (2.1) has no solution when t=n-1. But we know that the equation (2.1) has solution when t=n. For example

$X_{n\times n}=A_{n\times n}$

and

$$Y_{n\times n}=\begin{pmatrix} \max_j\{a_{1j}\} & 0 & \cdots & 0 \\ 0 & \max_j\{a_{2j}\} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \max_j\{a_{nj}\} \end{pmatrix}$$

is namely a set of the solutions of (2.1).

Thus from the theorem 1.1 it follows that $\rho s(A)=n$. Thereby from the theorem 2.2, we may know that A is full-rank.

Necessity. If A is full-rank fuzzy matrix, then $\rho s(A)=n$.

Thus from the theorem 1.1, it follows that formula (2.1) has no solution when t=n-1.

Theorem 2.3 Let $A\in\mathcal{M}_{n\times n}$, then A is full-rank iff A may be written as the sum of crros products, whose number is not less than n.

In fact, $\rho s(A)=n$ if A is fullrank. Thereby A may be expressed as the sum of n crros products.

On the contrary, if A may be expressed as the sum of crros products whose number is not less than n, then $\rho s(A)\geqslant n$; But $A\in\mathcal{M}_{n\times n}$, thus $\rho s(A)\leqslant n$. So $\rho s(A)=n$ certainly hold true. Finally from the theorem 2.1 it follows that A is full-rank.

3 INVERTIBLE FUZZY MATRIX

Definition 3.1 Let $A\in\mathcal{M}n\times n$, A is said to be an invertible matrix if there exists a fuzzy matrix $B\in\mathcal{M}n\times n$ making AB=BA=I hold true. B here is called inverse matrix of A.

Definition 3.2 For any $A\in\mathcal{M}n\times n$, A is called fuzzy permutation matrix if there exists just a single one thereas all the other elements are zero both in every row of A and in every column of A.

Proposition 3.1 Let $A\in\mathcal{M}n\times n$ be a fuzzy permutation matrix then A is namely the inverse matrix of A.

Proof Let $A=(a_{ij})\in\mathcal{M}n\times n$ and $AA^T=D=(d_{ij})\in\mathcal{M}n\times n$. Because there exists only one in every row of A, thereby

$$d_{ii}=\sum_{k=1}^{n}a_{ik}a_{ik}=1 \qquad (i=1,2,\ldots,n)$$

certainly hold true. Thus D=I. Similarily we may prove that $A^TA=I$.

Theorem 3.1 Let $A\in\mathcal{M}n\times n$, A is invertible iff A is a fuzzy permutation matrix.

Proof if A is a fuzzy permutation matrix, from the proposition 3.1 it comes that the A is namely inverse matrix of A, therefore A is an invertible fuzzy matrix.

On the contrary, there exists a $B=(b_{ij})\in\mathcal{M}n\times n$ making AB=BA=I hold true if A is an invertible matrix.

Suppose that A is not a fuzzy permutation matrix, then there exists three cases as follows:

Case 1 In some row of A (for example row one), there is a not only one but also a nonzero element. In order not to lose generality, we may let

$$A = \begin{bmatrix} 1 & a & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} \end{bmatrix} \qquad (0 < a \leq 1)$$

From AB=I, it follows that the elements of the first row of AB are definited by following equations

$$\begin{cases} 1\cdot b_{11} + a\cdot b_{21} = 1 \\ 1\cdot b_{12} + a\cdot b_{22} = 0 \\ \vdots \qquad \vdots \qquad \vdots \\ 1\cdot b_{1n} + a\cdot b_{2n} = 0 \end{cases} \qquad (3.1)$$

Solving for the set of equations (3.1), the matrix B will become

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ b_{21} & 0 & \cdots & 0 \\ b_{31} & b_{32} & \cdots & b_{3n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

Let now $D=(d_{ij})=BA=I$, it follows that $d_{12}=1\cdot a=0$. Hence $a=0$. It is in contradiction to the preceding supposition $0 < a \leq 1$.

Case 2 There's a not only one but also nonzero elements more than one in some row of A. Imitating the proof of the case 1, we may infer out similarily contradiction, too.

Case 3 There is no one in some row of A. For $\forall B \in \mathcal{M}_{n\times n}$, from calculating AB it follows that inequality $AB \neq I$ hold forever true if there is no one in some row of A.

Integrating above, it follows that there is only one and the other's are all zero in every row of A.

On the basis of the same reasons, we may prove that there is only one and the other's are all zero in every column of A. Therefore A is a fuzzy permutation matrix.

This theorem tell us that the invertible fuzzy matrices are namely boolean permutation matrices. Thus we have :

Theorem 3.2 In the set $\mathcal{M}_{n\times n}$, there are just n! invertible fuzzy matrices. And they are all fuzzy permutation matrices.

It is evident that invertible fuzzy matrices are all full-rank fuzzy matrices. Thereby we have yet

Theorem 3.3 The invertible fuzzy matrix is full-rank fuzzy matrix.

4 NONSINGULAR FUZZY MATRIX

Definition 4.1 Let $A\in\mathcal{M}_{m\times n}$, A is said to be a nonsingular fuzzy matrix if $\rho r(A)=m$ and $\rho c(A)=n$.

Thesis 4.1 Let $A\in\mathcal{M}_{m\times n}$, A is nonsingular iff both the row vectors and column vectors of A are all linear independent.

Theorem 4.1 Let $A\in\mathcal{M}_{n\times n}$, A is nonsingular iff $\rho r(A)=\rho c(A)=n$ and $\rho r(A)=\rho c(A)$ iff A is full-rank fuzzy matrix.

In the set $\mathcal{M}_{m\times n}$, the full-rank fuzzy matrix are distingctly nonsingular. But the nonsingular fuzzy matrix are not necessarily full-rank one. Thus nonsingular fuzzy matrix is a more extensive concept than the full-rank one. And yet in the set $\mathcal{M}_{n\times n}$, nonsingular fuzzy square matrix is equivalent to the full-rank fuzzy square matrix .

Thesis 4.2 If fuzzy square matrix A is invertible, then A is nonsingular one. If fuzzy square A is full-rank fuzzy matrix, then A is also nonsingular one.

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