MORE ON CARTESIAN PRODUCTS OVER INTUITIONISTIC FUZZY SETS

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Let E be a fixed set. Following the notations in [1] an IFS A in E is an object having the form:

$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \ / \ x \in E\}$,

where functions $\mu_A: E \longrightarrow [0, 1]$ and $\gamma_A: E \longrightarrow [0, 1]$ define the degree of membership and the degree of nonmembership of the element $x \in E$ to the set A, which is a subset of E and for every $x \in E$ $0 \le \mu_A(x) + \gamma_A(x) \le 1$.

For every IFS A the function $\pi_A(x) = 1 - \mu_A(x) - \gamma_A(x)$ defines the degree of indeterminancy.

For every two IFSs $A \subset E$ and $B \subset E$ the following relations are valid:

$A \subset B$, if $(\forall x \in E\ \mu_A(x) \le \mu_B(x)\ \&\ \gamma_A(x) \ge \gamma_B(x))$

$A = B$, if $(\forall x \in E\ \mu_A(x) = \mu_B(x)\ \&\ \gamma_A(x) = \gamma_B(x))$

For every IFS A and for every $\alpha, \beta \in [0, 1]$ the following operators are defined:

$D\alpha(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \gamma_A(x) + (1-\alpha).\pi_A(x)\rangle \ / \ x \in E\}$

$F\alpha\beta(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \gamma_A(x) + \beta.\pi_A(x)\rangle \ / \ x \in E\}$, $\alpha + \beta \le 1$

$G\alpha\beta(A) = \{\langle x, \alpha.\mu_A(x), \beta.\gamma_A(x)\rangle \ / \ x \in E\}$

$H\alpha\beta(A) = \{\langle x, \alpha.\mu_A(x), \gamma_A(x) + \beta.\pi_A(x)\rangle \ / \ x \in E\}$

$\bar{H}\alpha\beta(A) = \{\langle x, \alpha.\mu_A(x), \gamma_A(x) + \beta.(1 - \alpha.\mu_A(x) - \gamma_A(x))\rangle \ / \ x \in E\}$

$J\alpha\beta(A) = \{\langle x, \mu_A(x) + \alpha.\pi_A(x), \beta.\gamma_A(x)\rangle \ / \ x \in E\}$

$\bar{J}\alpha\beta(A) = \{\langle x, \mu_A(x) + \alpha.(1 - \mu_A(x) - \beta.\gamma_A(x)), \beta.\gamma_A(x)\rangle \ / \ x \in E\}$

$C(A) = \{\langle x, K, L\rangle \ / \ x \in E\}$, where $K = \max_{x \in E} \mu_A(x)$ & $L = \min_{x \in E} \gamma_A(x)$

$I(A) = \{\langle x, k, l\rangle \ / \ x \in E\}$, where $k = \min_{x \in E} \mu_A(x)$ & $l = \max_{x \in E} \gamma_A(x)$

$\Box(A) = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \ / \ x \in E\}$

$\Diamond(A) = \{\langle x, 1 - \gamma_A(x), \gamma_A(x)\rangle \ / \ x \in E\}$

Theorem 1. For every two IFSs $A \subset E$ and $B \subset E$ and for every operator $Y \in S = \{F\alpha\beta, D\alpha, G\alpha\beta, \bar{H}\alpha\beta, H\alpha\beta, \bar{J}\alpha\beta, J\alpha\beta, C, I, \Box, \Diamond\}$ when $A \subset B$ the relation $Y(A) \subset Y(B)$ is true.

Proof:

For $Y = F\alpha\beta$:

from

$\mu_{F\alpha\beta(A)}(x) = \mu_A(x) + \alpha.\pi_A(x) = \alpha + (1-\alpha).\mu_A(x) - \alpha.\gamma_A(x) \le$

$\leq \alpha + (1-\alpha).\mu_B(x) - \alpha.\gamma_B(x) = \mu_B(x) + \alpha.\pi_B(x) = \mu_{F\alpha\beta(B)}(x)$

and

$\gamma_{F\alpha\beta(A)}(x) = \gamma_A(x) + \beta.\pi_A(x) = \beta + (1-\beta).\gamma_A(x) - \beta.\mu_A(x) \geq$

$\geq \beta + (1-\beta).\gamma_B(x) - \beta.\mu_B(x) = \beta_B(x) + \beta.\pi_B(x) = \gamma_{F\alpha\beta(B)}(x)$

follows $F\alpha\beta(A) \subset F\alpha\beta(B)$.

Analogically for all other operators.

Let E1 and E2 be two universes and let

$$A = \{<x, \mu_A(x), \gamma_A(x)> / x \in E1\},$$
$$B = \{<y, \mu_B(y), \gamma_B(y)> / y \in E2\}$$

be two IFSs over them. Cartesian products over IFSs A and B are defined by:

$A \times_1 B = \{<x, \mu_A(x).\mu_B(y), \gamma\ (x).\gamma_B(y)> / x \in E1\ \&\ y \in E2\}$

$A \times_2 B = \{<x, \mu_A(x)+\mu_B(y)-\mu_A(x).\mu_B(y), \gamma_A(x).\gamma_B(y)>$
$/ x \in E1\ \&\ y \in E2\}$

$A \times_3 B = \{<x, \mu_A(x).\mu_B(y), \gamma_A(x)+\gamma_B(y)-\gamma_A(x).\gamma_B(y)>$
$/ x \in E1\ \&\ y \in E2\}$

$A \times_4 B = \{<x, \min(\mu_A(x),\mu_B(y)), \max(\gamma_A(x),\gamma_B(y))>$
$/ x \in E1\ \&\ y \in E2\}$

$A \times_5 B = \{<x, \max(\mu_A(x),\mu_B(y)), \min(\gamma_A(x),\gamma_B(y))>$
$/ x \in E1\ \&\ y \in E2\}$

Following definitions of cartesian products over IFSs and definitions of operators noted above next two theorems can be easily proved.

Theorem 2. For every two IFSs $A \subset E1$ and $B \subset E2$:

(a) $C(A\times_pB) = C(A)\times_pC(B)$, where $\times_p \in \{\times_1,\times_4,\times_5\}$

(b) $I(A\times_pB) = I(A)\times_pI(B)$, where $\times_p \in \{\times_1,\times_4,\times_5\}$

(c) $\Box(A\times_pB) = \Box(A)\times_p\Box(B)$, where $\times_p \in \{\times_2,\times_3,\times_4,\times_5\}$

(d) $\Diamond(A\times_pB) = \Diamond(A)\times_p\Diamond(B)$, where $\times_p \in \{\times_2,\times_3,\times_4,\times_5\}$

(e) $G\alpha\beta(A)\times_pG\alpha\beta(B) = G\alpha\beta(A\times_pB)$, where $\times_p \in \{\times_4,\times_5\}$

(f) $G\alpha\beta(A\times_1B) = G\alpha\beta(A)\times_1B = G\alpha\beta(B)\times_1A$

Theorem 3. For every two IFSs $A \subset E1$ and $B \subset E2$ and for every operator $Y \in \{F\alpha\beta, D\alpha, H\alpha\beta, \bar{H}\alpha\beta, J\alpha\beta, \bar{J}\alpha\beta\}$:

(a) $Y(A) \times_4 Y(B) \supset Y(A \times_4 B)$

(b) $Y(A) \times_5 Y(B) \subset Y(A \times_5 B)$

Let $A \subset E=E1\times E2\times\ldots\times En$ be an IFS and $q=(i_1,i_2,\ldots,i_K)$ be an index sequans, q' be the compliment of q whith respect to (1, 2, ...,n). Let $E(q)=Ei_1\times Ei_2\times\ldots\times Ei_K$ and E(q') be the compliment of

E(q) whith respect to E. Let $x(q)=(xi_1,xi_2,\dots xi_K)$ and $x(q')$ be the complement of $x(q)$ whith respect to $(x1,x2,\dots,xn)$.

Similarly to the projections of fuzzy sets introduced by Zadeh [2] the first and second projections of IFS A over E(q) will be noted with $A^1_{E(q)}$ and $A^2_{E(q)}$ and defined by:

$$A^1_{E(q)} = \{<(xi_1,xi_2,\dots xi_K),\ \max_{x(q')} \mu_A(x1,\dots xn),\ \min_{x(q')} \gamma_A(x1,\dots xn)> / (xi_1,\dots xi_K) \in E(q)\}$$

and

$$A^2_{E(q)} = \{<(xi_1,xi_2,\dots xi_K),\ \min_{x(q')} \mu_A(x1,\dots xn),\ \max_{x(q')} \gamma_A(x1,\dots xn)> / (xi_1,\dots xi_K) \in E(q)\}$$

It can be easily checked that first and second projections are still IFSs.

Let $A \subset E1xE2$ and $B \subset E1xE3$ be two IFSs. Following the notation in [3] of IFSs over different universes we shall define the intersection and the union of A and B by:

$$A \cap B = \{<(x,y),\ \min(\overline{\mu}_A(x,y),\overline{\mu}_B(x,y)),\ \max(\overline{\gamma}_A(x,y),\overline{\gamma}_B(x,y))> / x \in E1\ \&\ y \in E2 \cup E3\}$$

$$A \cup B = \{<(x,y),\ \max(\overline{\mu}_A(x,y),\overline{\mu}_B(x,y)),\ \min(\overline{\gamma}_A(x,y),\overline{\gamma}_B(x,y))> / x \in E1\ \&\ y \in E2 \cup E3\}$$

where:

$$\overline{\mu}_A(x,y) = \begin{cases} \mu_A(x,y) & y \in E2 \\ 0 & y \in E3 - E2 \end{cases},\quad \overline{\gamma}_A(x,y) = \begin{cases} \gamma_A(x,y) & y \in E2 \\ 1 & y \in E3 - E2 \end{cases}$$

and

$$\overline{\mu}_B(x,y) = \begin{cases} \mu_B(x,y) & y \in E3 \\ 0 & y \in E2 - E3 \end{cases},\quad \overline{\gamma}_B(x,y) = \begin{cases} \gamma_B(x,y) & y \in E3 \\ 1 & y \in E2 - E3 \end{cases}$$

For fixed $y \in E2 \cup E3$ we shall construct the IFSs A(y) and B(y) respectively as follows:

$$A(y) = \{<x,\ \overline{\mu}_A(x,y),\ \overline{\gamma}_A(x,y)> / x \in E1\ \&\ y \in E2 \cup E3,\ y \text{ is fixed}\}$$

$$B(y) = \{<x,\ \overline{\mu}_B(x,y),\ \overline{\gamma}_B(x,y)> / x \in E1\ \&\ z \in E2 \cup E3,\ y \text{ is fixed}\}$$

Theorem 4. (a) $A^1_{E1} = \bigcup_{y \in E2} A(y)$ & $A^2_{E1} = \bigcap_{y \in E2} A(y)$

(b) $(A \cap B)^1_{E1} = \bigcup_{y \in E2 \cup E3} (A(y) \cap B(y))$

(c) $(A \cup B)^2_{E1} = \bigcap_{y \in E2 \cup E3} (A(y) \cup B(y))$

Proof:

For (a):

from

$$\mu_{A^1_{E1}}(x) = \max_{y\in E2} \mu_A(x,y) = \max_{y\in E2} \mu_{A(y)}(x) = \mu_{\bigcup_{y\in E2} A(y)}(x)$$

and

$$\gamma_{A^1_{E1}}(x) = \min_{y\in E2} \gamma_A(x,y) = \min_{y\in E2} \gamma_{A(y)}(x) = \gamma_{\bigcup_{y\in E2} A(y)}(x)$$

follows $A^1_{E1} = \bigcup_{y\in E2} A(y)$.

(b) and (c) are proved analogically.

Let E1, E2, ..., En be n universes, let $E(q)=Ei_1 \times Ei_2 \times \ldots Ei_K$ and let $B \subset E(q)$ be an IFS. The cylindrical extention of the IFS B in the universe $E=E1\times E2\times\ldots\times En$ will be denoted cB and defined by:

$$cB=\{<(x_1,\ldots,x_n),\ \mu_{cB}(x_1,\ldots x_n),\ \gamma_{cB}(x_1,\ldots x_n)>/(x_1,\ldots x_n)\in E\}$$

where:

$$\mu_{cB}(x_1,\ldots x_n)=\mu_B(x_{i1},\ldots x_{iK}) \text{ and } \gamma_{cB}(x_1,\ldots x_n)=\gamma_B(x_{i1},\ldots x_{iK}),$$

where the $i_1, i_2, \ldots, i_K$ arguments of μ_{cB} and γ_{cB} are equal respectively to the first, second, ..., K^{th} arguments of μ_B and γ_B.

The IFS B will be called a base of the IFS cB.

Theorem 5. For every IFS $A \subset E=E1\times E2\times\ldots\times En$:

(a) $c(A^2_{E(q)}) \subset A \subset c(A^1_{E(q)})$

(b) $\bigcup^n_{i=1} c(A^2_{Ei}) \subset A \subset \bigcap^n_{i=1} c(A^1_{Ei})$

(c) $\bigcup^n_{i=1} c(A^2_{Ei}) = A^2_{E1} \times_5 A^2_{E2} \times_5 \ldots \times_5 A^2_{En}$

(d) $\bigcap^n_{i=1} c(A^1_{Ei}) = A^1_{E1} \times_4 A^1_{E2} \times_4 \ldots \times_4 A^1_{En}$

Proof:

For (a):

from

$$\mu_{c(A^2_{E(q)})}(x1,\ldots,xn) = \mu_{A^2_{E(q)}}(xi_1,\ldots,xiK) =$$
$$= \min_{x(q')} \mu_A(x1,\ldots,xn) \le \mu_A(x1,\ldots,xn) \le$$
$$\le \max_{x(q')} \mu_A(x1,\ldots,xn) = \mu_{c(A^1_{E(q)})}(x1,\ldots,xn)$$

and

$$\gamma_{c(A^2_{E(q)})}(x1,\ldots,xn) = \gamma_{A^2_{E(q)}}(xi_1,\ldots,xiK) =$$
$$= \max_{x(q')} \gamma_A(x1,\ldots,xn) \ge \gamma_A(x1,\ldots,xn) \ge$$

$$\geq \min_{x(q')} \gamma_A(x1,\ldots,xn) = \gamma_{c(A^1_{E(q)})}(x1,\ldots,xn)$$

follows $c(A^2_{E(q)}) \subset A \subset c(A^1_{E(q)})$.

(b) - (d) follows directly from (a) and from the definitions of union, intersection and cartesian products x_4 and x_5.

The IFS $A \subset E=E1xE2x\ldots xEn$ will be called (p,q)-separable if it can be present as x_p-cartesian products over its q-projections over Ei, i = 1,...n, $x_p \in \{x_1,x_2,x_3,x_4,x_5\}$ and $q \in \{1,2\}$.

Now it is easy to prove the validity of

Theorem 6. (a) If an IFS $A \subset E=E1xE2x\ldots En$ is (4,1)-separable, then every of its first projections will be (4,1)-separable.

(b) If an IFS $A \subset E=E1xE2x\ldots En$ is (5,2)-separable, then every of its second projections will be (5,2)-separable.

REFERENCES

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