LEVEL OPERATORS ON INTUITIONISTIC FUZZY SETS

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On the Intuitionistic Fuzzy Sets (IFS) [1] are defined several operators [2-4].

By analogy with [2-4] we shall define the following two operators (the definitions of the IFS-operations and operators are given in [1-5]):

$$!(A) = \{\langle x, \max(1/2, \mu_A(x)), \min(1/2, \gamma_A(x))\rangle / x \in E\},$$

$$?(A) = \{\langle x, \min(1/2, \mu_A(x)), \max(1/2, \gamma_A(x))\rangle / x \in E\},$$

which we shall call "level operators".

The assertions given below are valid for these operators.

THEOREM 1.14.1: For every IFS A:

(a) $\overline{!(\bar{A})} = ?(A)$;

(b) $?(A) \supset A \supset !(A)$;

(c) $!(?(A)) = ?(!(A)) = \{\langle x, 1/2, 1/2\rangle / x \in E\}$.

Proof: (a) $\overline{!(\bar{A})} = \overline{!\{\langle x, \gamma_A(x), \mu_A(x)\rangle / x \in E\}}$

$= \overline{\{\langle x, \max(1/2, \gamma_A(x)), \min(1/2, \mu_A(x))\rangle / x \in E\}}$

$= \{\langle x, \min(1/2, \mu_A(x)), \max(1/2, \gamma_A(x))\rangle / x \in E\}$

$= ?(A)$.

(c) $!(?(A)) = !\{\langle x, \min(1/2, \mu_A(x)), \max(1/2, \gamma_A(x))\rangle / x \in E\}$

$= \{\langle x, \max(1/2, \min(1/2, \mu_A(x))), \min(1/2, \max(1/2, \gamma_A(x)))\rangle / x \in E\}$

$= \{\langle x, 1/2, 1/2\rangle / x \in E\}$. ◊

THEOREM 1.14.2: For every two IFSs A and B:

(a) $!(A \cap B) = !(A) \cap !(B)$,

(b) $!(A \cup B) = !(A) \cup !(B)$,

(c) $?(A \cap B) = ?(A) \cap ?(B)$,

(d) $?(A \cup B) = ?(A) \cup ?(B)$.

Proof: (a) $!(A \cap B)$

$= \{\langle x, \max(1/2, \min(\mu_A(x), \mu_B(x))), \min(1/2, \max(\gamma_A(x), \gamma_B(x)))\rangle$

$/ x \in E\}$

$= \{\langle x, \min(\max(1/2, \mu_A(x)), \max(1/2, \mu_B(x))), \max(\min(1/2, \gamma_A(x)),$

$\min(1/2, \gamma_B(x))>/ x\in E\}$

$= !(A) \cap !(B).$ ◊

Similar equations for the operations "+", +." and "@" are not valid.

THEOREM 1.14.3: For every IFS A:

(a) $\Box!A = !\Box A$,

(b) $\Box?A = ?\Box A$,

(c) $\Diamond!A = !\Diamond A$,

(d) $\Diamond?A = ?\Diamond A$.

Proof: (a) $\Box!A = \{<x, \max(1/2, \mu_A(x)), 1 - \max(1/2, \mu_A(x))>/ x\in E\}$

$= \{<x, \max(1/2, \mu_A(x)), \min(1/2, 1-\mu_A(x))>/ x\in E\}$

$= !\Box A.$ ◊

It is checked directly, that there are not connections of the above form between XYp and YXp, where $X \in \{D_\alpha, F_{\alpha,\beta}, G_{\alpha,\beta}, H_{\alpha,\beta}, H^*_{\alpha,\beta}, J_{\alpha,\beta}, J^*_{\alpha,\beta}\}$ and $Y \in \{@, \$\}$.

The above defined two operators can be extended as follows:

$K_\alpha(A) = \{<x, \max(\alpha, \mu_A(x)), \min(\alpha, \gamma_A(x))>/ x\in E\}$,

$L_\alpha(A) = \{<x, \min(\alpha, \mu_A(x)), \max(\alpha, \gamma_A(x))>/ x\in E\}$,

where $\alpha \in [0, 1]$.

Obviously:

$$!(A) = K_{1/2}(A),$$

$$?(A) = L_{1/2}(A),$$

for every IFS A. The validity of the above theorems for the operators K_α and L_α is checked directly.

THEOREM 1.14.4: For every IFS A and for every $\alpha \in [0, 1]$:

(a) $\overline{K_\alpha(\bar{A})} = L_\alpha(A)$;

(b) $L_\alpha(A) \supset A \supset K_\alpha(A)$;

(c) $K_\alpha(L_\alpha(A)) = L_\alpha(K_\alpha(A)) = \{<x, \alpha, \alpha>/x\in E\}$.

THEOREM 1.14.5: For every two IFSs A and B:

(a) $K_\alpha(A \cap B) = K_\alpha(A) \cap K_\alpha(B)$,

(b) $K_\alpha(A \cup B) = K_\alpha(A) \cup K_\alpha(B)$,

(c) $L_{\alpha}(A \cap B) = L_{\alpha}(A) \cap L_{\alpha}(B)$,

(d) $L_{\alpha}(A \cup B) = L_{\alpha}(A) \cup L_{\alpha}(B)$.

Operators $P_{\alpha,\beta}$ and $Q_{\alpha,\beta}$ are modifications of the last two ones

$$P_{\alpha,\beta}(A) = \{<x, \max(\alpha, \mu_A(x)), \min(\beta, \gamma_A(x))>/ x \in E\},$$

$$Q_{\alpha,\beta}(A) = \{<x, \min(\alpha, \mu_A(x)), \max(\beta, \gamma_A(x))>/ x \in E\},$$

for α, $\beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Obviously,

$$K_{\alpha}(A) = P_{\alpha,\alpha}(A),$$

$$L_{\alpha}(A) = Q_{\alpha,\alpha}(A),$$

for every IFS A. The last operators satisfy:

THEOREM 1.14.6: For every IFS A and for every α, β, Γ, $\partial \in [0, 1]$, such that $\alpha + \beta \leq 1$, $\Gamma + \partial \leq 1$:

(a) $\overline{P_{\alpha,\beta}(\bar{A})} = Q_{\beta,\alpha}(A)$;

(b) $P_{\alpha,\beta}(Q_{\Gamma,\partial}(A)) = Q_{\max(\alpha,\Gamma),\ \min(\beta,\partial)}(P_{\alpha,\beta}(A))$;

(c) $Q_{\alpha,\beta}(P_{\Gamma,\partial}(A)) = P_{\min(\alpha,\Gamma),\ \max(\beta,\partial)}(Q_{\alpha,\beta}(A))$;

Proof: (b) $P_{\alpha,\beta}(Q_{\Gamma,\partial}(A))$

$= P_{\alpha,\beta}(\{<x, \min(\Gamma, \mu_A(x)), \max(\partial, \gamma_A(x))>/ x \in E\})$

$= \{<x, \max(\alpha, \min(\Gamma, \mu_A(x)), \min(\beta, \max(\partial, \gamma_A(x))>/ x \in E\}$

$= \{<x, \min(\max(\alpha, \Gamma), \max(\alpha, \mu_A(x))), \max(\min(\beta, \partial),$
$\max(\beta, \gamma_A(x)))>/ x \in E\}$

$= Q_{\max(\alpha,\Gamma),\ \min(\beta,\partial)}(\{<x, \max(\alpha, \mu_A(x)), \max(\beta, \gamma_A(x))>/ x \in E\})$

$= Q_{\max(\alpha,\Gamma),\ \min(\beta,\partial)}(P_{\alpha,\beta}(A))$. ◊

THEOREM 1.14.7: For every two IFSs A and for every α, $\beta \in [0, 1]$, such that $\alpha + \beta \leq 1$:

(a) $P_{\alpha,\beta}(A \cap B) = P_{\alpha,\beta}(A) \cap P_{\alpha,\beta}(B)$,

(b) $P_{\alpha,\beta}(A \cup B) = P_{\alpha,\beta}(A) \cup P_{\alpha,\beta}(B)$,

(c) $Q_{\alpha,\beta}(A \cap B) = Q_{\alpha,\beta}(A) \cap Q_{\alpha,\beta}(B)$,

(d) $Q_{\alpha,\beta}(A \cup B) = Q_{\alpha,\beta}(A) \cup Q_{\alpha,\beta}(B)$.

THEOREM 1.14.8: For every two IFSs A and for every $\alpha, \beta \in [0, 1]$, such that $\alpha + \beta \leq 1$:

(a) $C(P_{\alpha,\beta}(A)) = P_{\alpha,\beta}(C(A))$,

(b) $I(P_{\alpha,\beta}(A)) = P_{\alpha,\beta}(I(A))$,

(c) $C(Q_{\alpha,\beta}(A)) = Q_{\alpha,\beta}(C(A))$,

(d) $I(Q_{\alpha,\beta}(A)) = Q_{\alpha,\beta}(I(A))$.

Proof: (a) $C(P_{\alpha,\beta}(A))$

$= C(\{<x, \max(\alpha, \mu_A(x)), \min(\beta, \gamma_A(x))>/ x\in E\})$

$= \{<x, \max_{x\in E}(\max(\alpha, \mu_A(x))), \min_{x\in E}(\min(\beta, \gamma_A(x)))>/ x\in E\}$

$= \{<x, \max(\alpha, \max_{x\in E} \mu_A(x)), \min(\beta, \min_{x\in E} \gamma_A(x))>/ x\in E\}$

$= P_{\alpha,\beta}(C(A))$. ◊

From this theorem directly follows

THEOREM 1.14.9: For every two IFSs A and for every $\alpha, \in [0, 1]$:

(a) $C(K_\alpha(A)) = K_\alpha(C(A))$,

(b) $I(K_\alpha(A)) = K_\alpha(I(A))$,

(c) $C(L_\alpha(A)) = L_\alpha(C(A))$,

(d) $I(L_\alpha(A)) = L_\alpha(I(A))$.

(e) $C(!(A)) = !(C(A))$,

(f) $I(!(A)) = !(I(A))$,

(g) $C(?(A)) = ?(C(A))$,

(h) $I(?(A)) = ?(I(A))$.

A justification of the name ot this type operators is the following assertion (see § 1.4).

THEOREM 1.14.10: For every two IFSs A and for every $\alpha, \beta \in [0,1]$, such that $\alpha + \beta \leq 1$:

(a) $!(A) \in E^{1/2, 1/2}$,

(b) $K_\alpha(A) \in E^{\alpha,\alpha}$,

(c) $P_{\alpha,\beta}(A) \in E^{\alpha,\beta}$,

where:

$$E^{\alpha,\beta} = \{X/ (\forall x \in E)(\mu_X(x) \geq \alpha \ \& \ \gamma_X(x) \leq \beta\}$$

is a set from (α, β)-level, by analogy with the fuzzy set from α-level idea (e.g. [6]).

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