48

ALGREBRAIC PURISM IS EXPENSIVE IF IT CONCERNS FUZZY QUANTITIES

Milan Mareš
Prague

Calculations over fuzzy numbers and, more generally, over fuzzy quantities respect some more or less obvious rules. The tendency to define the basic arithmetic operations in the way sufficiently simulating the classical algebraic methods is natural and perhaps also acceptable. As shown in some of the referred papers (and in many others) the nature of fuzziness does not allow full transmission of standard crisp algebra to fuzzy quantities. It means that it is quite acceptable to ask which modifications (usually weakening) of the operations over fuzzy quantities imply the narrowing of the existing gap between fuzzy and crisp methods of numeric calculations. One approach to this problem was suggested and published by the author. This brief contribution is to summarize the general background of the approach and its consequences for understanding the difference between fuzzy and crisp numeric values and their computational properties.

1 Fuzzy Quantity – Basic Notions

In the whole paper we denote by R the set of all real numbers and by R_0 the set of all non-zero real numbers, $R_0 = R - \{0\}$.

Any fuzzy subset a of R with the membership function $f_a : R \to [0,1]$ will be called a *fuzzy quantity* iff

(1) $$\sup(f_a(x) : x \in R) = 1,$$

(2) $$\exists x_1 < x_2 \in R : f_a(x) = 0 \quad \text{for all } x \notin [x_1, x_2].$$

Both axioms are natural. Condition (1) is usually fulfilled in the stronger form

$$\exists x \in R : f_a(x) = 1,$$

and condition (2) means the natural demand of limitedness of the support set of f_a, i.e. the limitedness of the area of possible values of the fuzzy quantity a. Nevertheless, some of the referred papers show the possibility of avoiding even those weak assumptions without very essential loss of the derived results.

We denote by $\mathbb{R}$ the set of all fuzzy quantities fulfilling (1) and (2). It is useful to denote also

(3) $$\mathbb{R}_0 = \{a \in \mathbb{R} : \exists \varepsilon > 0, \forall x \in (-\varepsilon, \varepsilon), f_a(x) = 0\},$$

(4) $$\mathbb{R}^+ = \{a \in \mathbb{R}_0 : \forall x \leq 0, f_a(x) = 0\},$$

(5) $$\mathbb{R}^- = \{a \in \mathbb{R}_0 : \forall x \geq 0, f_a(x) = 0\},$$

(6) $$\mathbb{R}^* = \mathbb{R}^+ \cup \mathbb{R}^-.$$

If $a \in \mathbb{R}$ then it is useful to denote $-a \in \mathbb{R}$ where

$$f_{-a}(x) = f_a(-x) \quad \text{for all } x \in R. \tag{7}$$

If $a \in \mathbb{R}_0$ then we denote by $1/a$ the fuzzy quantity in $\mathbb{R}_0$ for which

$$f_{1/a}(x) = f_a(1/x) \quad \text{for all } x \in R_0, \tag{8}$$

$$f_{1/a}(0) = 0. \tag{9}$$

The definition of $\mathbb{R}_0$ by (3) instead of simpler form $\{a \in \mathbb{R} : f_a(0) = 0\}$ (cf. [7]) is given by the respect to (2) demanded for both, a and $1/a$.

If $y \in R$ then we denote by $\langle y \rangle \in \mathbb{R}$ the fuzzy quantity for which

$$f_{\langle y \rangle}(y) = 1, \qquad f_{\langle y \rangle}(x) = 0 \quad \text{iff } x \neq y. \tag{10}$$

For a, $b \in \mathbb{R}$ the equality $a = b$ means $f_a(x) = f_b(x)$ for all $x \in R$.

2 Arithmetic operations

The general Representation theorem (cf. [1,5]) can be used to define the basic arithmetic operations of addition and multiplication.

If a, $b \in \mathbb{R}$ then the fuzzy quantity $a \oplus b \in \mathbb{R}$ with membership function

$$f_{a \oplus b}(x) = \sup_{y \in R} (\min(f_a(y), f_b(x-y)) = \sup_{z \in R} (\min(f_a(x-z), f_b(z))) \tag{11}$$

is called the *sum of a and b*. Further, the fuzzy quantity $a \odot b$ with membership function

$$\begin{aligned} f_{a \odot b}(x) &= \sup_{y \in R_0} (\min(f_a(y), f_b(x/y)) = \\ &= \sup_{z \in R_0} (\min(f_y(x/z), f_b(z))), \quad \text{for } x \neq 0, \\ f_{a \odot b}(0) &= \max(f_a(0), f_b(0)), \end{aligned} \tag{12}$$

is called the *product of a and b* [1,7,10,11].

If $a \in \mathbb{R}$ and $r \in R$ then the product of crisp r and fuzzy a, denoted by $r \cdot a$, is defined by a simplification of (11), namely $r \cdot a = \langle r \rangle \odot a$, where for $x \in R$

$$\begin{aligned} f_{r \cdot a}(x) &= f_a(x/r) \quad \text{for } r \neq 0 \\ &= f_{\langle 0 \rangle}(x) \quad \text{for } r = 0. \end{aligned} \tag{13}$$

The addition and multiplication defined over fuzzy quantities possess some (but not all) of the classical algebraic properties.

3 Monoidal Properties – Why not More?

It can be easily verified [1,7,10] that the set $\mathbb{R}$ of fuzzy quantities forms a *commutative monoid* regarded to operations $\oplus$ and $\odot$. It means that for a, b, $c \in \mathbb{R}$

$$a \oplus b = b \oplus a, \qquad a \odot b = b \odot a, \tag{14}$$

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c, \qquad a \odot (b \odot c) = (a \odot b) \odot c, \tag{15}$$

$$a \oplus \langle 0 \rangle = a, \qquad a \odot \langle 1 \rangle = a. \tag{16}$$

It is also easily verifiable that the remaining group properties, namely

$$a \oplus (-a) = \langle 0 \rangle \qquad \text{and} \qquad a \odot (1/a) = \langle 1 \rangle \tag{17}$$

cannot be generally fulfilled.

This fact is in certain sense evident as the missing group properties demand the existence of strict equality between result of operations over fuzzy quantities (i. e. also a fuzzy quantity) and a crisp number. It contradicts to the nature of fuzziness. Wishing to avoid this contradiction it is potentially possible to respect the following two principles. First, to substitute some kind of "fuzzy zero", or "fuzzy unit" for crisp right hand sides $\langle 0 \rangle$ or $\langle 1 \rangle$ in (16), respectively. Second, to introduce some weaker form of similarity or equivalence instead of strict equalities in (16). Such weaker relation could be more adequate to the vague nature of fuzzy values entering (16). The equivalences suggested in [7,8,10] satisfy both of these principles. Their basic concept is briefly remembered in section 5, 6 and also 7.

4 One-Sided Distributivity

The product of crisp and fuzzy quantities $r \cdot a$, $r \in R$, $a \in \mathbb{R}$, is distributive in the sense that for $r \in R$, $a, b \in \mathbb{R}$,

$$r \cdot (a \oplus b) = (r \cdot a) \oplus (r \cdot b). \tag{18}$$

The complementary distributivity

$$(r_1 + r_2) \cdot a = (r_1 \cdot a) \oplus (r_2 \cdot a) \tag{19}$$

is not generally fulfilled. It means, for example, that generally

$$a \oplus a \neq 2 \cdot a, \quad a \in \mathbb{R}.$$

This declination of the traditional arithmetic habits cannot be easily explained by the absence of "fuzzy zero" or by too exaggerating demands of strict equality. Hence, the methods mentioned below and motivated by the intention to find some weaker validity of (16) do not imply the validity of (19).

The problem is much more essential. The multiplication of some quantity by a natural number n cannot be generally the same like the n-times repeated addition – except the quantity is crisp. In the more general case of fuzzy quantities with their vagueness the proper meaning of $\odot$ (or of its special case (12)) is too different from the essential nature of repeating the addition $\oplus$.

5 Symmetry

The remaining group property of the addition $\oplus$ over $\mathbb{R}$, namely

$$a \oplus (-a) = \langle 0 \rangle, \tag{20}$$

cannot be fulfilled, as it demands strict equality between fuzzy and crisp quantities. In accordance with the heuristic conclusions presented in Section 3, the following method was suggested in [10] and in [8, 9].

If $y \in R$ then $s \in \mathbb{R}$ such that for any $x \in R$

$$f_s(y + x) = f_s(y - x) \tag{21}$$

is called *y-symmetric*. By $\mathbb{S}_y$ we denote the set of all y-symmetric fuzzy quantities, and by $\mathbb{S}$ we denote the union

$$\mathbb{S} = \bigcup_{y \in R} \mathbb{S}_y.$$

We say that $a, b \in \mathbb{R}$ are *additively equivalent* and write $a \sim_{\oplus} b$, iff there exist $s_1, s_2 \in \mathbb{S}_0$ such that

$$a \oplus s_1 = b \oplus s_2. \tag{22}$$

It is not difficult to verify that for $a = b$ also $a \sim_{\oplus} b$ and, moreover,

$$a \oplus (-a) \sim_{\oplus} \langle 0 \rangle, \quad \text{i.e} \quad a \oplus (-a) \in \mathbb{S}_0. \tag{23}$$

These facts, together with the additive parts of (13), (14), (15), mean that the addition $\oplus$ over $\mathbb{R}$ is a commutative group operation if the strict equality is substituted by the additive equivalence meaning that fuzzy quantities are equivalent (or "sufficiently similar") iff they differ only in "fuzzy zeros" from $\mathbb{S}_0$.

6 Transversibility

Analogously to the additive case we may proceed also in the multiplicative one. Nevertheless the specific properties of multiplication, namely those connected with the zero-element and with the conversion of signs in case of multiplication by negative numbers, cause some serious difficulties [7, 8].

To avoid them we have to limit the analogous procedure to fuzzy quantities from $\mathbb{R}^*$ defined by (6), using (4) and (5).

If $y \in R_0$, $a \in \mathbb{R}^*$ then we say that a is *y-transversible* iff for

$$f_a(y \cdot x) = f_a(y/x) \quad \text{for all } x \neq 0. \tag{24}$$

The set of all y-transversible fuzzy quantities is denoted by $\mathbb{T}_y$, and by $\mathbb{T}$ we denote the union

$$\mathbb{T} = \bigcup_{y \in R_0} \mathbb{T}_y.$$

We say that fuzzy quantities $a, b \in \mathbb{R}^*$ are *multiplicatively equivalent*, and write $a \sim_{\odot} b$, iff there exist $t_1, t_2 \in \mathbb{T}_1$ such that

$$a \odot t_1 = b \odot t_2. \tag{25}$$

It can be easily verified [7, 8, 12] that $a = b$ implies $a \sim_{\odot} b$ for $a, b \in \mathbb{R}$ and, moreover, if $a \in \mathbb{R}^*$ then

$$a \odot (1/a) \sim_{\odot} \langle 1 \rangle \quad \text{i.e.} \quad a \odot (1/a) \in \mathbb{T}_1. \tag{26}$$

It means that the multiplication operation $\odot$ defines a commutative group on the set $\mathbb{R}^*$ if the strict equality is substituted by the multiplicative equivalence.

It should be remembered, too, that for $s \in \mathbb{S}_0$ and $a \in \mathbb{R}$ the product $s \odot a$ always belongs to $\mathbb{S}_0$. Even in this sense elements from $\mathbb{S}_0$ play the role of "fuzzy zero".

7 Linearity of Symmetric Fuzzy Quantities

The equivalences remembered above cannot generally solve the absence of the distributivity (18). Anyhow, for a special type of fuzzy quantities some weak form of distributivity can be derived.

The set $\mathbb{S}$ of symmetric fuzzy quantities is a commutative group according to the addition $\oplus$. Moreover, not only (17) is true but for $s \in \mathbb{S}$ and $r_1, r_2 \in R$ also

$$(r_1 + r_2) \odot s \sim_{\oplus} (r_1 \cdot s) \oplus (r_2 \cdot s). \tag{27}$$

It means that the set $\mathbb{S}$ of symmetric fuzzy quantities is a linear space if the additive equivalence $\sim_{\oplus}$ is considered instead of the strict equality.

The effectivity of this result is rather limited. It means nothing else but the fact that the linear operations (addition and multiplication by crisp real number) over symmetric fuzzy quantities do not affect the symmetry. In other, more heuristic, formulation – if we limit the operations to crisp numbers contaminated by 0-symmetric fuzzy noise [11, 6], and if we (through the equivalence $\sim_{\oplus}$) ignore the differences limited to 0-symmetric "fuzzy zeros", then the properties of linear space can be fully exploited.

8 Uniform Approach – An Open Problem

The combined application of addition $\oplus$ and multiplication $\odot$ keeps an open problem. Their distributivity is an exceptional phenomenon concerning very special cases only [1].

An attempt to construct some universal equivalence concept [8], having the properties of $\sim_{\oplus}$ in the case of addition, and the analogous properties of $\sim_{\odot}$ in the case of multiplication, did not offer fully acceptable solution. Its universality implies too large equivalence classes, and it does not influence the distributivity properties.

In this sense the construction of some equivalence relation applicabled to both, addition and multiplication, as well as to their combinations, keeps an open problem.

9 Conclusions

It is evident that the consistency of arithmetic operations, so advantageous in the case of crisp quantities, is not fully transferable to the fuzzy ones. The purpose of the research briefly remembered here was to test the conditions under which the transfer is, at least partly, possible.

Obviously, the extension of algebraic methods to fuzzy numbers must be compensed by some loss of other pleasant properties, and the question to be answered is, what we have to pay for approximate similarity between algebras of crisp and fuzzy quantities.

This compensation is natural and it corresponds to the essence of the matter in the case of addition. The additive equivalence, respecting the fuzziness of the input and ignoring its symmetric component, is evidently adequate to the natural features of the considered phenomena.

The multiplication of fuzzy quantities is rather more complicated. The weakening of the operations analogous to the additive case is effective only for some fuzzy quantities (for those from $\mathbb{R}^*$), meanwhile for the others the application of that method causes serious difficulties [7].

The full distributivity of the multiplication by crisp number can be by the same method guaranteed for a very narrow class of symmetric fuzzy quantities only.

Summarizing those facts it is acceptable to conclude that the algebraic elegancy familiar in the crisp case is for fuzzy values achieved for an often enormous price or it is not achievable at all.

References

[1] D. Dubois, H. Prade: Fuzzy numbers: An overview. In: Analysis of Fuzzy Information. Ed. James C. Bezdek, CRC Press, Boca Raton 1988, Vol. 1, 3–39.

[2] G. J. Klir, T. A. Folger: Fuzzy Sets, Uncertainty and Information. Prentice Hall, Englewood Cliffs - London 1988.

[3] M. Kovács, L. H. Tran: Algebraic structure of centered M-fuzzy numbers. Fuzzy Sets and Systems, 39 (1991), 3, 91–100.

[4] P. Lubczonok: Fuzzy vector spaces. Fuzzy Sets and Systems, 38 (1990), 3, 329–343.

[5] U. Höhle: Representation theorem for L-fuzzy quantities. Fuzzy Sets and Systems 3 (1981), 1, 83–107.

[6] M. Mareš: Space of symmetric normal fuzzy quantities. Int. Journal of General Systems, submitted.

[7] M. Mareš: Multiplication of fuzzy quantities. Kybernetika 28 (1992), 5, 337–356.

[8] M. Mareš: Algebraic equivalences over fuzzy quantities. Kybernetika, to appear.

[9] M. Mareš: Additive decomposition of fuzzy quantities with finite supports. Fuzzy Sets and Systems, 47 (1992), 3, 341–346.

[10] M. Mareš: Addition of fuzzy quantities: Disjunction–conjunction approach. Kybernetika 25 (1989), 2, 104–116.

[11] M. Mareš: Fuzzy data processing. Fuzzy Sets and Systems, submitted.

[12] M. Mareš: Remarks on fuzzy quantities with finite support. Kybernetika, submitted.

RNDr. Milan Mareš, DrSc., ÚTIA ČSAV (Institute of Information Theory and Automation – Czech Academy of Sciences), P. O. Box 18, 182 08 Praha 8, Czechoslovakia.