Random Fuzzy Sets and Fuzzy Martingales

Li Lushu
Department of Mathematics, Huaiyin Teachers' College
Jiangsu 223001, P.R.China

Abstract: We study random fuzzy sets and their relationship to fuzzy set-valued measures in a separable Banach space. Using the conditional expectations of random fuzzy sets, we introduce the concept of fuzzy martingales. Some properties and convengence theorems of fuzzy martingales are investigated.

Keywords: Propability space, random fuzzy set, conditional expectation, fuzzy martingale.

1. Introduction

The concept of fuzzy random variables was introduced by Puri and Ralescu [6] on the basis of the set representation of fuzzy sets. It enables us to use the rich mathematical apparatus of the theory of random sets and set-valued measures. The definitions and properties developed by Puri and Ralescu [12] allows us to further develop the concepts of random fuuzy sets in a Banach space. The purpose of this paper is to study the conditional expectations of random fuzzy sets and fuzzy martingales.

2. Random Sets and Random Fuzzy Sets

Throughout this paper, (Ω, Σ, P) will be a complete probability space, where the probability measure P is nonatomic. Let X be a separable Banach space with norm $\|.\|$, and let $K(X)$ and $CoK(X)$ denote the family of all nonempty compact and nonempty compact convex subsets of X, respectively. A linear structure in $K(X)$ is defined by

$$A + B = \{ a + b: a\in A, b\in B \} \quad \text{and} \quad \lambda A = \{ \lambda a: a \in A \}$$

The topology in $K(X)$ is introduced via the Hausdorff distance

$$d_H(A, B) = \max\{ \sup_{a\in A}\inf_{b\in B} \| a - b \|, \sup_{b\in B}\inf_{a\in A} \| a - b \| \}$$

The norm of $A\in K(X)$ is defined as $\| A \|_H = \sup_{a\in A} \| a \|$. A random set is a Σ-measurable set-valued mapping $f: \Omega \to K(X)$. For a random set f, let $S(f)$ be the set of integrable selectors of f. Then the Aumann integral of f is defined by (see [2], [3])

$$(A) \int f dP = \{ \int \varphi \, dP; \ \varphi \in S(f) \}$$

A random set f is called integrable bounded if $\int \|f\|_H dP < \infty$. Note that because the prob. measure P is nonatomic, $\int \|f\|_H dP < \infty$ implies the existance of $\int f \, dP \in CoK(X)$. More details on the measurability and integrability of random sets can be found in [3, 5, or 8].

Let $F^*(X)$ denote the family of all fuzzy sets $\mu : X \to [0, 1]$ with the properties

(a) μ is uppercontinuous

(b) $L_\alpha(\mu)$ is non-empty compact and convex for each $\alpha \in [0, 1]$.

where $L_\alpha(\mu)$ is the μ-level set of α defined via

$$L_\alpha(\mu) = \begin{cases} \{ x \in X; \mu(x) \geq \alpha \} & \text{if } \alpha > 0 \\ cl\{ x \in X; \mu(x) > 0 \} & \text{if } \alpha = 0 \end{cases}$$

A linear structure in $F^*(X)$ is defined by the operations

$$(\mu + \nu)(x) = \sup_{0 \leq \alpha \leq 1} \{\alpha ; \ x \in L_\alpha(\mu) + L_\alpha(\nu) \}$$

$$(\lambda \mu)(x) = \sup_{0 \leq \alpha \leq 1} \{\alpha ; \ x \in \lambda L_\alpha(\mu) \}$$

for $\mu, \nu \in F^*(X)$ and $\lambda \in K$. The metric in $F^*(X)$ is defined by

$$\delta(\mu, \nu) = \sup_{0 \leq \alpha \leq 1} d_H(L_\alpha(\mu), L_\alpha(\nu))$$

and the norm $\|\mu\|$ of a fuzzy set $\mu \in F^*(X)$ is defined as

$$\|\mu\| = \sup_{0 \leq \alpha \leq 1} \|L_\alpha(\mu)\|_H.$$

For μ_n (n>1), $\mu \in F^*(X)$, we denote $d_H(L_\alpha(\mu_n), L_\alpha(\mu)) \to 0$ and $\delta(\mu_n, \mu) \to 0$ by $\mu_n \xrightarrow{\alpha} \mu$ and $\mu_n \xrightarrow{\delta} \mu$, respectively.

Definition 2.1. A random fuzzy set is a mapping $\mu : \Omega \to F^*(X)$ such that $L_\alpha(\mu)$ is a random set for each $\alpha \in [0, 1]$.

Definition 2.2. The expected value of random fuzzy set μ, denoted by $E\mu$, is the fuzzy set such that

$$E\mu(x) = \sup_{0 \leq \alpha \leq 1}\{\alpha ; \ x \in (A) \int L_\alpha(\mu(\omega))dP \} \quad (x \in X)$$

Definition 2.3. A random fuzzy set μ is called integrably bounded if the random set $L_0(\mu)$ is integrably bounded. The sequence of random fuzzy sets $\{\mu_n\}$ is called uniformly integrably bounded if the sequence of random sets $\{L_0(\mu_n)\}$ is uniformly integrably bounded.

Note that the existence and uniqueness of $E\mu$ for an integrably bounded random fuzzy set μ are established in [6], and we have

$$L_\alpha(E\mu) = (A)\int L_\alpha(\mu(\omega))dP \qquad (\text{ Aumann's integral })$$

Applying the propertieas of Aumann's integrals [2] [3], we get

Theorem 2.1. Suppose μ, ν are integrably bounded random fuzzy sets. Then (1) $E\mu \in F^{**}(X)$

(2) $\delta(E\mu, E\nu) < \int \delta(\mu(\omega), \nu(\omega))\, dP$

(3) $E(\mu + \nu) = E\mu + E\nu$

Proof. (1) See [6, theorem 4.2].

(2) Since $E\mu$, $E\nu \in F^{**}(X)$, we have

$$\delta(E\mu, E\nu) = \sup_{0\leq\alpha\leq 1} d_H(L_\alpha(E\mu), L_\alpha(E\nu))$$
$$= \sup_{0\leq\alpha\leq 1} d_H((A)\int L_\alpha(E\mu(\omega))dP, (A)\int L_\alpha(E\nu(\omega))dP)$$
$$< \sup_{0\leq\alpha\leq 1} \int d_H(L_\alpha(E\mu(\omega)), L_\alpha(E\nu(\omega)))dP$$
$$< \int [\sup_{0\leq\alpha\leq 1} d_H(L_\alpha(E\mu(\omega)), L_\alpha(E\nu(\omega)))]dP$$
$$= \int \delta(\mu(\omega), \nu(\omega))dP$$

(3) For every $x \in X$, we have

$$= \sup_{0\leq\alpha\leq 1} \{\alpha ; x \in (A)\int L_\alpha(\mu(\omega))dP + (A)\int L_\alpha(\nu(\omega))dP\}$$
$$= \sup_{0\leq\alpha\leq 1} \{\alpha ; x \in L_\alpha(\mu) + L_\alpha(\nu)\} = (E\mu + E\nu)(x)$$

i.e. $E(\mu + \nu) = E\mu + E\nu$ Q.E.D.

Theorem 2.2. Let μ_n $(n>1)$, μ be a sequence of uniformly integrably bounded random fuzzy sets. Then

(1) for each $\alpha \in [0, 1]$,

$\mu_n(\omega) \to \mu(\omega)$ a.e. ====> $E\mu_n(\omega) \to E\mu(\omega)$ a.e.

(2) $\mu_n(\omega) \to \mu(\omega)$ a.e. ====> $E\mu_n(\omega) \to E\mu(\omega)$ a.e.

Proof. (1) For each $\alpha \in [0, 1]$, $L_\alpha(\mu_n)$ $(n>1)$, $L_\alpha(\mu)$ is a sequence of uniformly integrably bounded convex random sets. It follows from [8] that $d_H((A)\int L_\alpha(\mu_n(\omega))dP, (A)\int L_\alpha(\mu(\omega))dP) \to 0$ a.e., that is $E\mu_n \to E\mu$ a.e.

(2) Follows from theorem 2.1.(2) immediately. Q.E.D.

Definition 2.2. Let Σ_o be a sub-σ-algebra of Σ and μ an integrably bounded random fuzzy set. The coditional expectation of μ with respect to Σ_o, denoted by $E(\mu/\Sigma_o)$, is the random fuzzy set with properties:

(a) $E(\mu/\Sigma_o)$ is Σ_o-measurable

(b) $E(E(\mu/\Sigma_o).\chi_A) = E(\mu.\chi_A)$ for every $A\in\Sigma_o$

Note that the existance and uniqueness (P-a.e) of $E(\mu/\Sigma_o)$ are established in [7] and [8]. Some of the properties of this conditional expectation are stated next:

Theorem 2.3. Let μ, ν be integrably bounded random fuzzy sets. Then

(1) $L_\alpha(E\mu/\Sigma_o)) = E(L_\alpha(\mu)/\Sigma_o)$ for each $\alpha\in[0, 1]$

(2) $E((\alpha\mu+\beta\nu)/\Sigma_o) = \alpha E(\mu/\Sigma_o) + \beta E(\nu/\Sigma_o)$

(3) $E(E(\mu/\Sigma_o)) = E(\mu)$

(4) if μ is Σ_o-measurable, then $E(\mu/\Sigma_o) = \mu$ a.e.

(5) if $\Sigma_1\subset\Sigma_2$ are two sub-σ-algebras of Σ, then

$$E(E(\mu/\Sigma_1)/\Sigma_2) = E(\mu/\Sigma_1) = E(E(\mu/\Sigma_2)/\Sigma_1)$$

Proof. (1) See [7, proposition 4.1].

(2) (3) (4) can be verify easily.

(5) $\Sigma_1\subset\Sigma_2$ implies that $E(\mu/\Sigma_1)$ is Σ_2-measureable. It follows from (4) that $E(E(\mu/\Sigma_1)/\Sigma_2) = E(\mu/\Sigma_1)$ a.e. For every $A\in\Sigma_1\subset\Sigma_2$, since $E(E(\mu/\Sigma_2)\chi_A)=E(\mu\chi_A)$, we have $E(E\mu/\Sigma_2)/\Sigma_1) = E(\mu/\Sigma_1)$. This copletes the proof of (5). Q.E.D.

Theorem 2.4. Let μ_n $(n\geq 1)$, μ be a sequence of integrably bounded random fuzzy sets. Then

(1) for each $\alpha\in[0, 1]$,

$$\mu_n \xrightarrow{\alpha} \mu \text{ a.e} \Longrightarrow E(\mu_n/\Sigma_o) \xrightarrow{\alpha} E(\mu/\Sigma_o) \text{ a.e.}$$

(2) $$\mu \xrightarrow{\delta} \mu \text{ a.e} \Longrightarrow E(\mu_n/\Sigma_o) \xrightarrow{\delta} E(\mu/\Sigma_o) \text{ a.e.}$$

Proof. Follows from theorem 2.2 evidently.

3. Fuzzy Martingales and Their Convergence

In this section, let $N = \{0, 1, 2, 3 \ldots\}$ and $\overline{N} = N \cup \{\infty\}$. Let $\{\Sigma_n, n\in N\}$ be an increasing sequence of sub-σ-algebras of Σ, and $\Sigma_\infty = \sigma(\bigcup_{n\geq 1}\Sigma_n) = \Sigma$. The sequence $\{\mu_n, \Sigma_n; n\in N\}$ of random fuzzy sets and increasing sub-σ-algebras will be called a adaptive random fuzzy process if $\mu_n:\Omega \to F^m(X)$ is a Σ_n-measurable integrably bounded random fuzzy set for each $n\in N$. A random variable $\tau: \Omega \to \overline{N}$ is called a stopping time if $[\tau = n] = \{\omega\in\Omega; \tau(\omega) = n\}\in\Sigma_n$ for each $n\in N$. For a stopping time τ, we define a σ-algebra Σ_τ as

$$\Sigma_\tau = \{A\in\Sigma; A\cap[\tau = n] \in\Sigma_n, n\in N\}$$

Definition 3.1. A fuzzy submartingale (supermartingale) is a adaptive random fuzzy process $\{\mu_n, \Sigma_n; n\in N\}$ such that, for $\forall$ m, n with m<n,

$$E(\mu_n/\Sigma_m) > \mu_m \quad (E(\mu_n/\Sigma_m) < \mu_m) \quad \text{a.e.}$$

If $\{\mu_n, \Sigma_n; n\in N\}$ is both a submartingale and a supermartingale, it is called a fuzzy martingale.

Obviously, $\{\mu_n, \Sigma_n; n\in N\}$ is a fuzzy submartingale (supermartingale) iff $\{L_\alpha(\mu_n), \Sigma_n; n\in N\}$ is a set-valued submartingal (supermartingale) for each $\alpha \in [0, 1]$ (see [4, 5, 8]).

To investigate the properties of fuzzy martingales, we only consider the case of fuzzy submartingale because there are corresponding results for fuzzy supermartingales.

Theorem 3.1. Suppose $\{\mu_n, \Sigma_n; n\in N\}$ is a fuzzy submartingale, s and t are two finite stopping time with $s < t$. Then

$$E(\mu_t/\Sigma_s) > \mu_s \quad \text{a.e.}$$

To prove this theorem, we need the following lemmas:

Lemma 3.1. Suppose $\{\mu_n, \Sigma; n\in N\}$ is a adaptive random fuzzy process, μ is a random fuzzy set. If we define $\mu_\infty = \mu$, then for any stopping time τ, μ_τ is a Σ_τ-measurable random fuzzy set.

Lemma 3.2. Suppose $\{f_n, \Sigma_n; n\in N\}$ is a set-valued submartingale, s and t are two stopping time with $s < t$. Then

$$E(f_t/\Sigma_s) \supseteq f_s \quad \text{a.e.}$$

We omit the proofs of lemma 3.1 and lemma 3.2.

Proof of Theorem 3.1. It follows from lemma 3.1 that μ_t is Σ_t-measurable and μ_s is Σ_s-measurable for the finite stopping time s and t. We also can show that μ_t and μ_s are both integrably bounded. So we have

$$\mu_s(x) = \sup_{0\leq\alpha\leq 1}\{\alpha ; x \in L\alpha(\mu_s)\}$$

and $$E(\mu_t/\Sigma_s)(x) = \sup_{0\leq\alpha\leq 1}\{\alpha ; x \in E(L\alpha(\mu_t))/\Sigma_s)\}$$

But for each $\alpha \in [0, 1]$, $\{L\alpha(\mu_n), \Sigma_n; n \in N\}$ is a set submartingale. By lemma 3.2, we get $E(L\alpha(\mu_t)/\Sigma_s) \supseteq L\alpha(\mu_s)$ a.e. Hence we have

$$E(\mu_t/\Sigma_s) \geq \mu_s \quad \text{a.e.}$$ Q.E.D.

Now we consider the convergence of fuzzy martingales. First, we discuss the δ-convergence, we have

Theorem 3.2. Let $\{\mu_n, \Sigma_n; n \in N\}$ be a fuzzy submartingale and μ an integrably bounded random fuzzy set. If $\| \mu_n - \mu \|_L \to 0$, then

$$\mu_n \xrightarrow{\delta} \mu \quad \text{a.e.}$$

Where the norm $\| . \|_L$ is the $L^1(\Omega, P; X)$-norm.

Proof. Since $\| \mu_n - \mu \|_L \to 0$, we know that for $\forall \varepsilon > 0$ and $\forall \sigma > 0$, there exists natural number N such that for $\forall n > N$,

$$\| \mu_n - \mu \|_L < \varepsilon \sigma$$

But $$P\{\omega \in \Omega ; \sup_{n>N} \| \mu_n - \mu \| > \sigma \}$$
$$\leq \lim \sup_{n>N} 1/\sigma . \int_{S\sigma} \| \mu_n - \mu \| dP$$
$$\leq \sup_{n>N} 1/\sigma . \| \mu_n - \mu \|_L = 1/\sigma . \sup_{n>N} \| \mu_n - \mu \|_L$$

where $S\sigma = \{\omega \in \Omega ; \sup_{n>N} \| \mu_n - \mu \| > \sigma \}$. Hence,

$$P\{\omega \in \Omega ; \sup_{n>N} \| \mu_n - \mu \| > \sigma \}$$
$$\leq 1/\sigma . \sup_{n>N} \| \mu_n - \mu \|_L \leq 1/\sigma . \varepsilon \sigma = \varepsilon .$$

That is $$\mu_n \xrightarrow{\delta} \mu \quad \text{a.e.}$$ Q.E.D.

Finally, we discuss the α-convergence of fuzzy martingales:

Theorem 3.3. Suppose $\{\mu_n, \Sigma_n; n \in N\}$ is a fuzzy submartingale. If $\sup_{n>1} E \| \mu_n(\omega) \|_S < \infty$, then there exists an integrably bounded random fuzzy set $\mu : \Omega \to F^*(X)$ such that $\mu_n \xrightarrow{\alpha} \mu$ a.e. and

$$E(\mu/\Sigma_n) \geq \mu_n \quad \text{a.e.} \quad (\forall n \in N)$$

Proof. For each $\alpha \in [0, 1]$, it follows from the conditions of this theorem that $\{L\alpha(\mu_n), \Sigma_n; n \in N\}$ is a set-ualued submartingale and

$\sup_{n>1} E\|L_\alpha(\mu_n)\|_H < \infty$. Using [8, theorem 9.5.8], we know that there exists an integrably bounded random set $f_\alpha : \Omega \rightarrow coK(X)$ such that

$$d_H(L_\alpha(\mu_n), f_\alpha) \rightarrow 0 \quad \text{a.e.}$$

Since $\alpha < \beta$ ====> $L_\alpha(\mu_n) \supseteq L_\beta(\mu_n)$, we have $\alpha < \beta$ ====> $f_\alpha \supseteq f_\beta$, i.e. f_α $(0 \leq \alpha \leq 1)$ are nested. If we let

$$\mu(\omega)(x) = \sup_{0 \leq \alpha \leq 1} \{ \alpha ; x \in f_\alpha(\omega) \}$$

then $\mu : \Omega \rightarrow F^*(X)$ is an integrably bounded random fuzzy set and

$$L_\alpha(\mu)(\omega) = f_\alpha(\omega) \qquad (\forall \alpha \in [0, 1])$$

Hence $d_H(L_\alpha(\mu_n), L_\alpha(\mu)) \rightarrow 0$ a.e. $(\forall \alpha \in [0, 1])$. That is

$$\mu_n \xrightarrow{\alpha} \mu \quad \text{P-a.e}$$

Further, for $\forall$ t, $n \in N$ with $n < m$, since $\{ \mu_n, \Sigma_n; n \in N \}$ is a fuzzy martingale, we have

$$E(\mu_m / \Sigma_n) > \mu_n$$

Let $m \rightarrow \infty$, it follows theorem 2.4 that

$$E(\mu / \Sigma_n) > \mu_n \quad \text{a.e.}$$

Q.E.D.

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