EXTENSION OF ADDITIVE FUZZY MEASURE

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Abstract: In this paper, we study the additive fuzzy measure which defined on generated fuzzy algebra, the extension theorem of normal additive fuzzy measure is given.

Key words: Generated fuzzy algebra, Normal additive fuzzy measure, Extension of additive fuzzy measure.

1. Introduction

The fuzzy measure discussed in this paper was originally introduced by Zadeh in [10], and then E.P.Klement[1],[2] did lots of researches in this field. In this paper, we mainly discuss the extension theorem of additive fuzzy measure.

In section 2 of this paper, the concept of class of fuzzy sets generated by classical class of sets is given, and we study its main properties. Then the monotone class theorem of fuzzy sets is proved. In section 3, we extend additive fuzzy measure from algebra $\tilde{\mathscr{F}}$ to the σ-algebra $\sigma(\tilde{\mathscr{F}})$ generated by $\tilde{\mathscr{F}}$ and prove that the extension is unique.

Throughout this paper, let X be nonempty, $\mathscr{F}(X)$ be the class of fuzzy sets, R be $[0,+\infty)$.

2. Class of fuzzy sets.

In his paper [11], Zadeh introduced the set-theoretic operations of fuzzy sets, such as union, intersection, and complement.

Definition 2.1. For $\tilde{A},\tilde{B}\in\mathscr{F}(X)$, $\tilde{A},\tilde{B}$ is called disjoint iff for every $x\in X$, we have $\tilde{A}(x)\wedge\tilde{B}(x)=0$; Sequence of fuzzy-sets $\tilde{A}_n$; $n\geq 1$ is called disjoint iff for arbitrary i, j, $i\neq j$, $\tilde{A}_i$, $\tilde{A}_j$ is disjoint.

Definition 2.2 Let $\{\tilde{A}_n;\ n\geq 1\}\subset\mathscr{F}(X)$, $\{\tilde{A}_n;\ n\geq 1\}$ is called an increasing sequence of fuzzy sets iff $\tilde{A}_1\subset\tilde{A}_2\subset\cdots$. Let $\tilde{A}=\bigcup_{n=1}^{\infty}\tilde{A}_n$

then $\widetilde{A}$ is called the limit of $\{A_n ; n \geq 1\}$ and we write $\widetilde{A}_n \uparrow \widetilde{A}$; Similarly if $\widetilde{A}_1 \supset \widetilde{A}_2 \supset \cdots$ and $\widetilde{A} = \bigcap_{n=1}^{\infty} \widetilde{A}_n$, we say that the $\widetilde{A}_n$ form a decreasing sequence of fuzzy sets and $\widetilde{A}$ is the limit of the $\widetilde{A}_n$, denoted by $\widetilde{A}_n \downarrow \widetilde{A}$.

Definition 2.3 Let $\widetilde{\mathfrak{F}} \subset \mathfrak{F}(X)$, we say that $\widetilde{\mathfrak{F}}$ is a fuzzy algebra iff

1) $X \in \widetilde{\mathfrak{F}}$.
2) For every $\widetilde{A} \in \widetilde{\mathfrak{F}} \Longrightarrow \widetilde{A}^c \in \widetilde{\mathfrak{F}}$.
3) For arbitrary $\widetilde{A}, \widetilde{B} \in \widetilde{\mathfrak{F}} \Longrightarrow \widetilde{A} \cup \widetilde{B} \in \widetilde{\mathfrak{F}}$.

From the definition 2.3, we obtain :

Proposition 2.1 Let $\widetilde{A}, \widetilde{B} \in \widetilde{\mathfrak{F}}$ then $\widetilde{A} \cap \widetilde{B}$, $\widetilde{A} - \widetilde{B} = \widetilde{A} \cap \widetilde{B}^c \in \widetilde{\mathfrak{F}}$.

Definition 2.4 Let $\widetilde{\mathcal{A}} \subset \mathfrak{F}(X)$, $\widetilde{\mathcal{A}}$ is called a fuzzy σ-algebra iff

1) $X \in \widetilde{\mathcal{A}}$.
2) For $\widetilde{A} \in \widetilde{\mathcal{A}} \Longrightarrow \widetilde{A}^c \in \widetilde{\mathcal{A}}$.
3) For $\{\widetilde{A}_n ; n \geq 1\} \subset \mathfrak{F}(X) \Longrightarrow \bigcup_{n=1}^{\infty} A_n \in \widetilde{\mathcal{A}}$.

If $\widetilde{\mathcal{A}}$ is a fuzzy σ-algebra, then $(X , \widetilde{\mathcal{A}})$ is called a fuzzy measurable space.

For a given fuzzy algebra $\widetilde{\mathfrak{F}}$, denote

$$\sigma(\widetilde{\mathfrak{F}}) = \bigcap \{\widetilde{\mathcal{A}} ; \widetilde{\mathfrak{F}} \subset \widetilde{\mathcal{A}} \text{ and } \widetilde{\mathcal{A}} \text{ is a fuzzy } \sigma\text{-algebra}\} .$$

It is easy to prove that $\sigma(\widetilde{\mathfrak{F}})$ is the smallest fuzzy σ-algebra containing $\widetilde{\mathfrak{F}}$, we say that $\sigma(\widetilde{\mathfrak{F}})$ is the fuzzy σ-algebra generated by $\widetilde{\mathfrak{F}}$.

Definition 2.5 A class of fuzzy sets $\widetilde{\mathcal{M}}$ is called a fuzzy monotone class iff for arbitrary monotone sequences of fuzzy sets $\{\widetilde{A}_n ; n \geq 1\}$, $\widetilde{A}_n \uparrow \widetilde{A}$ or $\widetilde{A}_n \downarrow \widetilde{A}$, we have $\widetilde{A} \in \widetilde{\mathcal{M}}$.

Theorem 2.1 (Monotone class theorem of fuzzy sets)

Let $\widetilde{\mathfrak{F}}$ be a fuzzy algebra, $\widetilde{\mathcal{M}}$ be a monotone class of fuzzy set containing $\widetilde{\mathfrak{F}}$, then $\sigma(\widetilde{\mathfrak{F}}) \subset \widetilde{\mathcal{M}}$.

Definition 2.6 Let $\mathcal{B}$ be a classical class of sets, for $\widetilde{A} \in \mathfrak{F}(X)$, we call that $\widetilde{A}$ is the fuzzy set generated by $\mathcal{B}$, if for arbitrary $\alpha \in [0, 1]$, we have: $A_\alpha, A_{\underset{\cdot}{\alpha}} \in \mathcal{B}$.

Let $\mathfrak{F}(\mathcal{B})$ denote the class of fuzzy sets generated by $\mathcal{B}$.

Proposition 2.2 1): If $\mathcal{B}$ is a algebra , then $\mathfrak{F}(\mathcal{B})$ is a fuzzy algebra. 2): If $\mathcal{B}$ is a σ-algebra, then $\mathfrak{F}(\mathcal{B})$ is a fuzzy σ-algebra, and in the same time, $A_\alpha \in \mathcal{B}$ and $A_{\underset{\cdot}{\alpha}} \in \mathcal{B}$ is eqivalent.

Definition 2.7 A fuzzy algebra $\widetilde{\mathfrak{F}}$ is called generatable iff there is a classical algebra $\mathfrak{F}_0$, such that $\widetilde{\mathfrak{F}} = \mathfrak{F}(\mathfrak{F}_0)$. and at the same time, $\widetilde{\mathfrak{F}}$ is called a fuzzy algebra generated by $\mathfrak{F}_0$; Fuzzy σ-algebra $\widetilde{\mathcal{A}}$ is called generatable iff there is a classical σ-algebra $\mathcal{A}$, such that

$\tilde{\mathscr{A}} = \mathfrak{F}(\mathscr{A})$. And at the same time $\tilde{\mathscr{A}}$ is called a fuzzy σ-algebra generated by $\mathscr{A}$.

Proposition 2.3 Let $\mathfrak{F} = \mathfrak{F}(\mathfrak{F}_0)$ be a generable fuzzy algebra , then for arbitrary $A \in \mathfrak{F}_0$, $\alpha \in [0, 1]$, we have that $\alpha I_A \in \mathfrak{F}$.

Proposition 2.4 Let $(X, \mathscr{A})$ be a measurable space , $\tilde{\mathscr{A}}$ be a fuzzy σ-algebra, then $\tilde{\mathscr{A}}$ is generated by $\mathscr{A}$ iff for all $\tilde{A} \in \tilde{\mathscr{A}}$, $\tilde{A}$ is a measurable function on $(X, \mathscr{A})$.

3 . Additive fuzzy measure on class of fuzzy sets and its extension.

Definition 3.1 Let $\mathfrak{F}$ be a fuzzy algebra , the fuzzy set function $\mu : \mathfrak{F} \longrightarrow R^+$ is called an additive fuzzy measure on $\mathfrak{F}$, if it satisfies the following conditions:

AFM1: $\mu(\phi)=0$.

AFM2: If $\tilde{A}, \tilde{B} \in \mathfrak{F}$, and $\tilde{A} \subset \tilde{B}$, then $\mu(A) \leq \mu(B)$.

AFM3: If $\tilde{A}, \tilde{B} \in \mathfrak{F}$, then $\mu(\tilde{A} \cup \tilde{B}) + \mu(\tilde{A} \cap \tilde{B}) = \mu(\tilde{A}) + \mu(\tilde{B})$.

AFM4: If $\{\tilde{A}_n;\ n \geq 1\} \subset \mathfrak{F}$, $\tilde{A} \in \mathfrak{F}$ and $\tilde{A}_n \uparrow \tilde{A}$, then $\lim_{n\to\infty} \mu(\tilde{A}_n) = \mu(\tilde{A})$. (A) .

AFM5: If $\{\tilde{A}_n;\ n \geq 1\} \subset \mathfrak{F}$, $\tilde{A} \in \mathfrak{F}$ and $\tilde{A}_n \downarrow \tilde{A}$, then $\lim_{n\to\infty} \mu(\tilde{A}_n) = \mu(\tilde{A})$.

If fuzzy set function μ only satisfis AFM1, AFM2, AFM3 , AFM4, then μ is called lower additive fuzzy measure ; If μ satisfies AFM1, AFM2, AFM3, AFM5, then μ is called upper additive fuzzy measure .

Obviously, μ is finitely additive and countably additive.

Definition 3.2 Let $\mathfrak{F} = \mathfrak{F}(\mathfrak{F}_0)$ be a generatable fuzzy algebra , an additive fuzzy measure μ on $\mathfrak{F}$ is called normal iff for every $A \in \mathfrak{F}_0$, $\alpha \in [0, 1]$, we have that $\mu(\alpha I_A) = \alpha \mu(I_A)$.

In the following , let $\mathfrak{F}$ be a fuzzy algebra, μ be an additive fuzzy measure on $\tilde{\mathfrak{F}}$.

Let $\tilde{\mathfrak{F}}_{\sigma}^{+} = \{ \tilde{A};\ A \in \mathfrak{F}(X)$, and there exist $\tilde{A}_n \in \tilde{\mathfrak{F}}$, $n \geq 1$, such that $\tilde{A}_n \uparrow A \}$.

$\tilde{\mathfrak{F}}_{\sigma}^{-} = \{ A;\ \tilde{A} \in \tilde{\mathfrak{F}}(X)$, and there exist $\tilde{A}_n \in \tilde{\mathfrak{F}}$, $n \geq 1$, such that $\tilde{A}_n \downarrow A \}$

Proposition3.1 For all $\tilde{A} \in \tilde{\mathfrak{F}}(X)$, $\tilde{A} \in \tilde{\mathfrak{F}}_{\sigma}^{+}$ iff $\tilde{A}^c \in \tilde{\mathfrak{F}}_{\sigma}^{-}$.

Proposition3.2 $\tilde{\mathfrak{F}}_{\sigma}^{+}$ has the following structural characteristics:

1) $\tilde{\mathfrak{F}} \subset \tilde{\mathfrak{F}}_{\sigma}^{+}$.

2) If $\tilde{A}$, $\tilde{B} \in \tilde{\mathfrak{F}}_{\sigma}^{+}$, then $\tilde{A} \cup \tilde{B}$, $\tilde{A} \cap \tilde{B} \in \tilde{\mathfrak{F}}_{\sigma}^{+}$.

3) If $\tilde{A}_n \in \tilde{\mathfrak{F}}_{\sigma}^{+}$, $n \geq 1$, $\tilde{A}_n \uparrow \tilde{A}$, then $\tilde{A} \in \tilde{\mathfrak{F}}_{\sigma}^{+}$.

Lemma 3.1 Let μ be an additive fuzzy measure on $\tilde{\mathfrak{F}}$, suppose that $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ ... belong to $\tilde{\mathfrak{F}}$ and increase to a limit $\tilde{A}$; $\tilde{B}_1$, $\tilde{B}_2$, $\tilde{B}_3$... belong to $\tilde{\mathfrak{F}}$ and increase to $\tilde{B}$. If $\tilde{A} \subset \tilde{B}$, then

$$\lim_{n\to\infty} \mu(\tilde{A}_n) \leq \lim_{n\to\infty} \mu(\tilde{B}_n) .$$

Theorem3.1 Let μ be an additive fuzzy measure on $\tilde{\mathfrak{F}}$,then μ can be extended to a lower additive fuzzy measure $\bar{\mu}$ on $\tilde{\mathfrak{F}}_{\sigma}^{+}$. Where $\bar{\mu}$ is defined by the following manner:

For every $\tilde{A} \in \tilde{\mathfrak{F}}_{\sigma}^{+}$, then there exist $\tilde{A}_n \in \tilde{\mathfrak{F}}$, such that $\tilde{A}_n \uparrow \tilde{A}$, we define: $\bar{\mu}(\tilde{A}) = \lim_{n\to\infty} \mu(\tilde{A}_n)$.

Theorem3.2 Let $\tilde{\mathfrak{F}} = \tilde{\mathfrak{F}}(\mathfrak{F}_0)$ be a fuzzy algebra generated by $\mathfrak{F}_0$, μ is a normal additive fuzzy measure on $\tilde{\mathfrak{F}}$, then there exists a measure ν on $\sigma(\mathfrak{F}_0)$ such that for every $\tilde{A} \in \tilde{\mathfrak{F}}$, $\tilde{A}(x)$ is ν-integrable, and $\mu(\tilde{A}) = \int_X \tilde{A}(x)\, d\nu$.

Corollary 1 If $\tilde{\mathfrak{F}} = \tilde{\mathfrak{F}}(\mathfrak{F}_0)$ is a generated fuzzy algebra , μ is a normal additive fuzzy measure on $\tilde{\mathfrak{F}}$, then μ satisfies:

$$\mu(\tilde{A}) + \mu(\tilde{A}^c) = \mu(X) \text{ , wherever } \tilde{A} \in \tilde{\mathfrak{F}} .$$

Corollary 2 If $\tilde{\mathfrak{F}} = \tilde{\mathfrak{F}}(\mathfrak{F}_0)$ is a generated fuzzy algebra, μ is a normal additive fuzzy measure on $\tilde{\mathfrak{F}}$, $\tilde{A}_n$ $(n \geq 1)$ is an increasing sequence of fuzzy sets on $\tilde{\mathfrak{F}}$, $\tilde{B}_m$ $(m \geq 1)$ is a decreasing sequence of fuzzy sets on $\tilde{\mathcal{F}}$, and $\bigcup_{n=1}^{\infty} A_n \supset \bigcap_{m=1}^{\infty} B_m$, then $\lim_{n\to\infty}\mu(\tilde{A}_n) \geq \lim_{m\to\infty}\mu(\tilde{B}_m)$.

Lemma 3.2 Let $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}(\tilde{\mathcal{F}}_0)$ be a generated fuzzy algebra , μ be a normal additive fuzzy measure on $\tilde{\mathcal{F}}$, then for every $\tilde{A} \in \tilde{\mathcal{F}}_\sigma^+$, we have:

$$\bar{\mu}(\tilde{A}) + \underline{\mu}(\tilde{A}^c) = \mu(X).$$

Where $\underline{\mu}$ is a fuzzy set function on $\tilde{\mathcal{F}}_\sigma^-$, which defined by the following manner:

For $\tilde{A} \in \tilde{\mathcal{F}}_\sigma^-$, then there exist $\tilde{A}_n$, $n\geq 1$, such that $\tilde{A}_n \downarrow \tilde{A}$, the fuzzy set function $\underline{\mu} : \tilde{\mathcal{F}}_\sigma^- \to R^+$ is defined by (3.2),

$$\underline{\mu}(\tilde{A}) = \lim_{n\to\infty} \mu(\tilde{A}_n) \tag{3.2}$$

Theorem3.3 Let $\bar{\mu}$ be a lower additive fuzzy measure on $\tilde{\mathcal{F}}_\sigma^+$, which is defined by (3.1), for $\tilde{A} \in \tilde{\mathcal{F}}(X)$, we define:

$$\mu^*(\tilde{A}) = \inf\{\ \bar{\mu}(\tilde{G}),\ \tilde{G} \in \tilde{\mathcal{F}}_\sigma^+,\ G \supset A\ \}$$

Then;

1) $\mu^* = \bar{\mu}$ on $\tilde{\mathcal{F}}_\sigma^+$, and μ^* is a finite fuzzy set function.

2) If $\tilde{A} \subset \tilde{B}$, then $\mu^*(\tilde{A}) \subset \mu^*(\tilde{B})$.

3) If $\tilde{A}$, $\tilde{B} \in \tilde{\mathcal{F}}(X)$, then $\mu^*(\tilde{A} \cup \tilde{B}) + \mu^*(\tilde{A} \cap \tilde{B}) \leq \mu^*(\tilde{A}) + \mu^*(\tilde{B})$.

4) If $\tilde{A}_n \in \tilde{\mathcal{F}}(X)$, $\tilde{A}_n \uparrow \bigcup_{n=1}^{\infty} \tilde{A}_n$, then $\lim_{n\to\infty} \mu^*(\tilde{A}_n) = \mu^*(\bigcup_{n=1}^{\infty} \tilde{A}_n)$.

Lemma3.3 If $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}(\tilde{\mathcal{F}}_0)$ is a generated fuzzy algebra, μ is a normal additive fuzzy measure on $\tilde{\mathcal{F}}$, then for $\tilde{A} \in \tilde{\mathcal{F}}(X)$, we have:

$$\mu^*(\tilde{A}) + \mu^*(A^c) \geq \mu(X).$$

Theorem3.4 If $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}(\tilde{\mathcal{F}}_0)$ is a generated fuzzy algebra, μ is a normal additive fuzzy measure on $\tilde{\mathcal{F}}$, denote

$$\tilde{\mathcal{A}} = \{\ \tilde{A} \in \tilde{\mathcal{F}}(X);\ \ \mu^*(\tilde{A}) + \mu^*(\tilde{A}^c) = \mu(X)\ \}.$$

Then $\tilde{\mathcal{A}}$ is a fuzzy σ-algebra containing $\tilde{\mathcal{F}}$.

Theorem3.5 If $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}(\tilde{\mathcal{F}}_0)$ is a generated fuzzy algebra, μ is a normal additive fuzzy measure on $\tilde{\mathcal{F}}$, then μ^* is an additive fuzzy measure on $\tilde{\mathcal{A}}$.

Theorem3.6 (Extension Theorem) If $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}(\tilde{\mathcal{F}}_0)$ is a generated fuzzy algebra, μ is a normal additive fuzzy measure on $\tilde{\mathcal{F}}$, then μ has a unique extension to an additive fuzzy measure on $\sigma(\tilde{\mathcal{F}})$.

Definition3.3 The additive fuzzy measure μ on $\hat{\mathcal{A}}$ is called complete iff whenever $\tilde{A}\in\hat{\mathcal{A}}$ and $\mu(\tilde{A}) = 0$ we have $\tilde{B}\in\hat{\mathcal{A}}$ for all $\tilde{B}\subset\tilde{A}$.

Theorem3.7 In the Theorem3.5 , the additive fuzzy measure μ^* on $\hat{\mathcal{A}}$ is complete.

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