# ON THE THEOREMS FOR λ-ADDITIVE FUZZY MEASURES

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In this paper, the relationship between probability measures and λ-additive fuzzy measures is used to give the definition of the product measurse of λ-additive fuzzy measures and Fubini's theorem is discussed. A Lebesgue decomposition-like theorem, kolmogorov's zero-one law and the important Borel-Cantelli lemma, for λ-additive fuzzy measures, are established.

1. Fuzzy integral

Let X be a nonempty set and let Σ be a σ-algebra on X. A set function μ from Σ to [0,1] is called a fuzzy measure on Σ [5] if it satisfies the following conditions:

(1) $\mu(\phi)=0,$ (1.1)

$\mu(X)=1;$

(2) if $A, B\in\Sigma$ and $A\subset B$, then $\mu(A)\leqslant\mu(B)$; (1.2)

(3) if $\{A_n\}$ is a monotone sequence of sets in Σ, then

$$\lim_{n\to\infty}\mu(A_n)=\mu(\lim_{n\to\infty}A_n). \tag{1.3}$$

Let $\lambda\in(-1,+\infty)$, $\lambda\neq 0$, be a real number. A fuzzy measure μ on Σ is said to a λ-additive fuzzy measure on Σ [5], if whenever $A, B\in\Sigma$ and $A\cap B=\phi$, then

$$\mu(A\cup B)=\mu(A)+\mu(B)+\lambda\mu(A)\mu(B) \tag{1.4}$$

holds.

We are able to prove that a fuzzy measure μ on Σ is a λ-additive fuzzy measure, if and only if, for every sequence $\{A_n\}$ of disjont sets in Σ, we have

$$\mu(\bigcup_{n=1}^{\infty}A_n)=\frac{1}{\lambda}\left[\prod_{n=1}^{\infty}(1+\lambda\mu(A_n))-1\right]$$

There exists a relationship between probability measures and λ-additive fuzzy measures.

If μ is a λ-additive fuzzy measure on Σ, then

$$\mu^*:=\log(1+\lambda\mu)/\log(1+\lambda) \tag{1.5}$$

is a probabilitity measure on Σ. Conversely, if $\mu^*$ is a probability measure on Σ, then

$$\mu:=-\frac{1}{\lambda}+\frac{1}{\lambda}(1+\lambda)^{\mu^*}$$

is a λ-additive fuzzy measure on Σ [2], [6].

Thus we can use the relation to give the definition of a fuzzy integral.

Let μ be a λ-additive fuzzy measure on Σ, and let $f: X\to[0,1]$ be a Σ-measurable function. For an arbitrary $A \in \Sigma$ we define

$$\int_A f d\mu := -\frac{1}{\lambda} + \frac{1}{\lambda}(1+\lambda)^{\int_A (1+\lambda f)/\log(1+\lambda) d\mu^*} \tag{1.6}$$

and name the quantity the $\mu$-integral of over A [3].

2. Product measures of $\lambda$-additive fuzzy measures

**Definition 1.** Let $(X_1,\Sigma_1)$ and $(X_2,\Sigma_2)$ be measurable spaces. Let $\mu$ and $v$ be $\lambda$-additive fuzzy measures on $\Sigma_1$ and $\Sigma_2$, respectively, with the same parameter $\lambda$. We say that $\lambda$-additive fuzzy measure on $\Sigma_1 \times \Sigma_2$

$$\mu \times v := -\frac{1}{\lambda} + \frac{1}{\lambda}(1+\lambda)^{\mu^* \times v^*} \tag{2.1}$$

is a product measure of $\lambda$-additive fuzzy measures $\mu$ and $v$, where $\mu^*$ and $v^*$, as in (1.5), are probability measures on $\Sigma_1$ and $\Sigma_2$, respectively, and $\mu^* \times v^*$ denotes product measure on $\Sigma_1 \times \Sigma_2$ in classical measure theory.

Suppose that $\Psi = \{A_1 \times A_2 | A_1 \in \Sigma_1, A_2 \in \Sigma_2\}$, then it is a semiring and $\Sigma_1 \times \Sigma_2$ is a $\sigma$-algebra generated by $\Psi$. If $A_1 \times A_2 \in \Psi$, then

$$\frac{\log(1+\lambda\mu \times v(A_1 \times A_2))}{\log(1+\lambda)} = \frac{\log(1+\lambda\mu(A_1))}{\log(1+\lambda)} \cdot \frac{\log(1+\lambda v(A_2))}{\log(1+\lambda)} \tag{2.2}$$

holds.

**Theorem 1.** If $\gamma$ is a $\lambda$-additive fuzzy measure on $\Sigma_1 \times \Sigma_2$ and satisfies (2.2), then $\gamma = \mu \times v$.

We now give a statement of Fubini's theorem for product measures which has been defined above (2.1).

**Theorem 2.** Suppose that $f$: $X_1 \times X_2 \to [0,1]$ is a $\Sigma_1 \times \Sigma_2$-measurable function and $A_1 \times A_2 \in \Psi$. then

(1) the integrals

$$\int_{A_2} f(x_1,x_2)dv \qquad \int_{A_1} f(x_1,x_2)d\mu.$$

are $\Sigma_1$-measurable function and $\Sigma_2$-measurable function, respectively.

(2) the following equation holds:

$$\int_{A_1 \times A_2} f(x_1,x_2)d\mu dv = \int_{A_1} d\mu \int_{A_2} f(x_1,x_2)dv = \int_{A_2} dv \int_{A_1} f(x_1,x_2)d\mu$$

3. Decomposition of $\lambda$-additive fuzzy measures

The following theorem presents a Lebesgue decomposition-like theorem for $\lambda$-additive fuzzy measures

**Theorem 3.** Let $\mu$. $v$ be $\lambda$-additive fuzzy measures on $\Sigma$, then there exist two set functions $v_c$, $v_s$ : $\Sigma \to [0,1]$ with the properties (1.1)–(1.4) such that

$$v = v_c + v_s + \lambda v_c v_s, \quad v_c << \mu, \quad v_s \perp \mu$$

and this decomposition is unique.

4. Independent fuzzy events

**Definition 2.** Let $\mu$ be a $\lambda$-additive fuzzy measure on $\Sigma$

(1) Let $A_1$, $A_2 \cdots$, $A_n \in \Sigma$, then we say that $A_1$, $A_2$, $\cdots$, $A_n$ are mutually $\mu$-inde-

pendent if, for $2 \leqslant m \leqslant n$ and $1 \leqslant i_2 \leqslant i_2 \leqslant \cdots \leqslant i_m \leqslant n$, we have

$$\mu(\bigcap_{k=1}^{m} A_{ik}) = -\frac{1}{\lambda} + \frac{1}{\lambda}(1+\lambda)^{\prod_{k=1}^{m} \log(1+\lambda\mu(A_{ik}))/[\log(1+\lambda)]^{m}}$$

(2) Let $\Theta$ be a index set and for arbitrary $\theta \in \Theta$, $A_\theta \in \Sigma$. Then we say that $\{A_\theta : \theta \in \Theta\}$ is a mutually $\mu$–independent collection if each of its finite subcollection is mutually $\mu$–independent.

(3) Let $\Theta$ be a index set and for arbitrary $\theta \in \Theta$, $\Psi_\theta \subset \Sigma$. Then we say that $\{\Psi_\theta : \theta \in \Theta\}$ is a mutually $\mu$–independent family if $\{A_\theta : A_\theta \in \Psi_\theta,\ \theta \in \Theta\}$, for each choice of $A_\theta$ from $\Psi_\theta$, is a mutually $\mu$–independent collection.

**Theorem** 4. Let $\{A_n;\ n \geqslant 1\}$ be a mutually $\mu$–independent collection, then for each $A \in \bigcap_{n=1}^{\infty} \sigma(A_n, A_{n+1} \cdots)$, $\mu(A)$ is either 0 or 1.

**Theorem** 5. Let $\{A_n;\ n \geqslant 1\}$ be a mutually $\mu$–independent collection. then $\mu(\overline{\lim_{n\to\infty}} A_n)$ is either 0 or 1.

**Theorem** 6. Let $\mu$ be a $\lambda$–additive fuzzy measure on $\Sigma$ and let $\{A_n\}$ be a sequence in $\Sigma$, we have the following results:

(1) If $\sum_{n=1}^{\infty} \mu(A_n) < +\infty$, then $\mu(\overline{\lim_{n\to\infty}} A_n) = 0$;

(2) If $\{A_n; n \geqslant 1\}$ is a mutually $\mu$–independent collection and $\sum_{n=1}^{\infty} \mu(A_n) = +\infty$

then $\mu(\overline{\lim_{n\to}} A_n) = 1$.

**Corollary.** Let $\mu$ be a $\lambda$–additive fuzzy measure on $\Sigma$ and $\{A_n;\ n \geqslant 1\}$ be a mutually $\mu$–independent collection in $\Sigma$, then

(1) $\mu(\overline{\lim_{n\to\infty}} A_n) = 0$, iff $\sum_{n=1}^{\infty} \mu(A_n) < +\infty$.

(2) $\mu(\overline{\lim_{n\to\infty}} A_n) = 1$, iff $\sum_{n=1}^{\infty} \mu(A_n) = +\infty$.

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