Study on the dual problems in the theory of fuzzy measure and fuzzy integral

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Abstract:

In this paper,we propose the dual proposition rules for fuzzy measure and fuzzy integral,and list the dual relations of the most important concepts and propositions in the theory of fuzzy measure and fuzzy integral.

§1. Introduction

Sugeno [1] first proposed the concepts of fuzzy measure and fuzzy integral and Ralescu/Adams[2] extended these concepts to extended real-valued cases. Wang[3,4] and Sun [7,8] introduced some asymptotic structures of fuzzy measures such as null-additivity,pseudo-null-additivity,autocontinuity,pseudo-autocontinuity,uniform autocontinuity and pseudo-autocontinuity,which show the connections among some kinds of fuzzy measures.Also, they proved a series of convergence theorems for fuzzy integral provided that the corresponding fuzzy measures have one kind of asymptotic structures.Furthermore,the asymptotic structures of null-additivity and autocontinuity of fuzzy measures have been used in the discussions in the aspects of Riesz theorem,Egoroff theorem,Lebesgue theorem ([3,4]),Levi-like theorem,Borel-Cantelli lemma and Lusin theorem ([12]) to fuzzy measure and fuzzy integral.

The relations among these sympototic structures have been done in many articles [3,4,7,9.10].In this paper,under the condition that the fuzzy measures are finite,we present the dual proposition rules for fuzzy measures,and based on these dual proposition rules for fuzzy measures,the dual proposition rules for fuzzy integral are presented as well.Some dual concepts and dual propositions for fuzzy measure and fuzzy integral are listed, and these dual concepts and dual propositions are presented and proved in [3,4,7,9,10] repectively (at least respectively in principle). The conclusions of dual concepts and dual propositions obtained in this paper reveal the inner relations between the corresponding concepts and propositions in the theory of fuzzy measure and fuzzy integral.

Throughout this paper, X is a non-empty set (universum), $\mathcal{F}$ is a σ-algbra of sets of X, $(X,\mathcal{F},\mu)$ is a fuzzy measure space. The concepts and notations not defined in this paper can be found in [3,4,7,9.10].

§2 Some assumptions

§2.1 The assumption to fuzzy measure

In this paper, we suppose that the fuzzy measures are finite. It is the essential unique assumption in this paper.

§2.2 The assumption to the range of fuzzy integral

In this paper, we assume that all fuzzy integrals are taken over X and denote $\int f d\mu$ by $\int f d\mu$ in short.

This assumption is based on the following considerations. When we discuss the fuzzy integral over $A\in\mathcal{F}$, we can regard that A is the whole space without any loss of generality. In this time, the fuzzy measure space is $(A, A\cap\mathcal{F}, \mu\mid A)$ when $\mu\mid A$ is the restriction of fuzzy measure μ which defined on measurable space $(X,\mathcal{F})$. It is obvious that when μ is a finite fuzzy measure, so is $\mu\mid A$.

§2.3 The assumption to measurable function

We assume that the measurable functions f(x) appeared in this paper satisfy that $0\leqslant f(x)\leqslant \mu(X), \forall x\in X$. The reasonability for this assumption is twofold: on the first place, for any finite fuzzy measure μ and any measurable function $f:X\to[0,+\infty]$, we have $\int f d\mu=\int [f\wedge\mu(x)]d\mu$ in which $f\wedge\mu(x)$ satisfies $0\leqslant f(x)\wedge\mu(x)\leqslant\mu(x), \forall x\in X$, and on the other hand, this assumption do not change the relations between (or among) measurable functions. For instance, $f\leqslant g \Longrightarrow f\wedge\mu(x)\leqslant g\wedge\mu(x)$, $f_n\uparrow(\text{or}\downarrow)f \Longrightarrow f_n\wedge\mu(x)\uparrow(\text{or}\downarrow)f\wedge\mu(x)$, $f_n\to f \Longrightarrow f_n\wedge\mu(x)\to f\wedge\mu(x)$ etc.

§2.4 The assumption for pseudo-null-additivity, pseudo-autocontinuity and pseudo-uniform autocontinuity

For the needs of convergence pseudo-almost everywhere and convergence pseudo-uniform autocontinuity

For the needs of convergence pseudo-almost everywhere and convergence pseudo-in fuzzy measure of measurable functions with repect to $A\in\mathcal{F}$,Wang[4] introdued the concepts of pseudo-null-additivity,pseudo-autocontinuity of a fuzzy measure with repect to $A\in\mathcal{F}$.Since the fuzzy integrals considered in this paper are taken on X,then it is not necessary that we discuss the pseudo-null-additivity,pseudo-autocontinuity of a fuzzy measure with respect to $A\neq X$.From now on,we assume that pseudo-null-additivity,pseudo-autocontinuity (and pseudo-uniform autocontinuity) of a fuzzy measure are with respect to X and at the same time we omit the phrase "with respect to X" in the discussion.

§2.5 Some notations

In this paper,we denote $M:=\{\mu;\mu$ is a finite fuzzy measure on measurable space $(x,\mathcal{F})\}$ and $F:=\{f;f:x\to[0,\mu(X)]$ is $\mathcal{F}$ measurable function$\}$.In addition, we use P(or $\bar{P}$) to stand for the word "proposition" and P:[...] (or $\bar{P}$[...]) for the content of the proposition P(or $\bar{P}$).

§3. Dual concepts and dual propositions for fuzzy measure

§3.1 Dual fuzzy measures

Lamada/Moral[6] presented the concept of dual fuzzy measures.The following concepts of dual fuzzy measures is the extension of the one of Lamada/Moral[6].

Let $\mu\in M$,we call $\bar{\mu}\in M$ the dual fuzzy measure of μ where $\bar{\mu}$ is defined via $\bar{\mu}(A)=\mu(x)-\mu(\bar{A})$. $\forall A\in\mathcal{F},\bar{A}:=X-A$.We can prove that μ is also the dual fuzzy measure of $\bar{\mu}$ and $\bar{\bar{\mu}}:=\overline{(\bar{\mu})}=\mu$, $\bar{\mu}(X)=\mu(X)$.

§3.2 The dual proposition rules for fuzzy measure

Let P be a proposition for fuzzy measure μ.If we replace $\mu(A)$ by $\mu(x)-\bar{\mu}(\bar{A})$ in P,then we can obtain a proposition for fuzzy measure $\bar{\mu}$.We call the new proposition is the dual proposition of P and denote it by $\bar{P}$.It is evident that P and $\bar{P}$ mentioned above satisfy that P is true <==> $\bar{P}$ is true.

Note 3.1. It is not difficult to know that a concept(or definition) is a special form of proposition.Therefore,in this paper,some times,we call the concepts, such as " $\mu \in M$ is autocontinuous", propositions,and,when we emphasize the concepts themself,we still call them concepts.

Note 3.2. To the above dual proposition rules for fuzzy measure,it is worth to point out that although the propositions P and $\bar{P}$ seem the same things,when the proposition P is with some arbitrary conditions of $A \in \mathcal{F}$,we can replace $\bar{A}$ by A in $\bar{P}$ and obtain a proposition which is equivalent to $\bar{P}$ and is not as the same as $\bar{P}$ in the description (see the following examples of dual propositions for fuzzy measure).

§3.3 On the dual problems for fuzzy measure

In the following,we list some dual concepts and propositions for fuzzy measure by using the dual proposition rules for fuzzy measure proposed in section 3.2.

§3.3.1 The duality between null-additivity and pseudo-null-additivity

Theorem 3.1 The dual proposition of P:[$\mu \in M$ is 0-add.] is P:[$\bar{\mu} \in M$ is p.0-add.]

Theorem 3.1 can be also described by another form:

Theorem 3.1' $\mu \in M$ is 0-add.iff $\bar{\mu} \in M$ is p.0-add..

Afterwards,when we point out that $\bar{P}$ is the dual proposition of the proposition P we invisibly show that P is true iff $\bar{P}$ is true.

The proof of theorem 3.1

[$\mu \in M$ is 0-add.]

$\Leftrightarrow$[$\forall A,B \in \mathcal{F}$, $\mu(B)=0 \Rightarrow \mu(A-B)=\mu(A)$] (see[3,Proposition 2])

$\Leftrightarrow$[$\forall A,B \in \mathcal{F}$, $\bar{\mu}(X)-\bar{\mu}(\bar{B})=0 \Rightarrow \bar{\mu}(X)-\mu(\bar{A}\cup B)=\bar{\mu}(x)-\mu(\bar{A})$](from the dual proposition rules for fuzzy measure)

$\Leftrightarrow$[$\forall A,B \in \mathcal{F}$, $\bar{\mu}(\bar{B})=\bar{\mu}(X) \Rightarrow \bar{\mu}(A\cup B)=\bar{\mu}(A)$]

$\Leftrightarrow$[$\bar{\mu} \in M$ is P.0-add.]

Theorem 3.1 tell us that null-additivity and pseudo-null-additivity are two dual concepts.

§3.3.2 The duality between autocontunuity and pseudo-autocontinuity

Theorem 3.2 (1) The dual proposition of P:[$\mu \in M$ is autoc. ↑] is $\bar{P}$:[$\bar{\mu} \in M$ is p.autoc. ↓]

(2) The dual proposition of P:[$\mu \in M$ is autoc. ↓] is $\bar{P}$:[$\bar{\mu} \in M$ is p.autoc. ↑].

We omit the proof of theorem 3.2 (and the following theorems of dual propositions), since the ideas of the proof is similar to the one of theorem 3.1.

From [10,Theorem 3.1] and [9,Theorem 3.1],we know that μ is autoc.<==> μ is autoc. ↑<==> μ is autoc. ↓, and μ is p.autoc.<==> μ is p.autoc. ↓ <==> μ is p.autoc. ↑, when the fuzzy measure μ is finite.Hence,Theorem 3.2 shows that autocontinuity and pseudo-autocontinuity are two dual concepts.

Theorem 3.3 The dual proposition of P:[$\mu \in M$ is uniformly autocontinuous] is $\bar{P}$:[$\bar{\mu} \in M$ is pseudo-uniform autocontinuous].

Theorem 3.3 reveals that the concepts of uniform autocontinuity and pseudo-uniform autocontinuity are two dual concepts.

§3.4 The dual concepts for the convergences of sequence of measurable functions

Let $f \in F$.Define $\bar{f} \in F$ via $\bar{f}(x)=\mu(x)-f(x)$, $x \in X$.

§3.4.1, §3.4.2 and §3.4.3 give some dual propositions for the convergences of sequences of measurable functions.

§3.4.1 The duality between almost everywhere convergence and pseudo-almost everywhere convergence

Theorem 3.4 The dual proposition of P:[$f_n \xrightarrow{a.e.\ \mu} f$] is $\bar{P}$:[$\bar{f}_n \xrightarrow{p.a.e.\ \bar{\mu}} \bar{f}$].

Note 3.3 We can easily prove that $\bar{f}_n \xrightarrow{p.a.e.\ \bar{\mu}} \bar{f}$ <==> $f_n \xrightarrow{p.a.e.\ \bar{\mu}} f$.But for the needs of section 4,we only give the former one in Theorem 3.4.The similar cases shall appear in Theorem 3.5 and Theorem 3.6.

§3.4.2 The duality between convergence in fuzzy mesaure and convergence pseudo-in fuzzy measure

Theorem 3.5 The dual proposition of P:[$f_n \xrightarrow{\mu} f$] is $\bar{P}$:[$\bar{f}_n \xrightarrow{p.\ \bar{\mu}} \bar{f}$]

§3.4.3 The duality between almost uniform convergence and pseudo-almost uniform convergence

Theorem 3.6 The dual proposition of P:[$f_n \xrightarrow{a.u.\ \mu} f$] is $\bar{P}$:[$\bar{f}_n \xrightarrow{p.a.u.\ \bar{\mu}} \bar{f}$].

Theorem 3.4, Theorem 3.5 and Theorem 3.6 show that convergence almost everywhere versus convergence pseudo-almost everywhere, convergence in fuzzy measure versus convergence pseudo-in fuzzy measure and almost uniform convergence versus pseudo-almost uniform convergence are three groups of dual concepts.

§3.5 Some dualities in Riesz theorem, Egoroff Theorem and Lebesgue theorem

Wang [4] extended the Riesz theorem, Egoroff theorem and Lebesgue theorem in the sense of classical measures to the ones of fuzzy measures. In the following we shall study the dual properties in these three theorems.

§3.5.1 On Riesz theorem ([4, Theorem 4.1])

Under the assumption of section 2 and by using the conclusions of 3.3, we can represent the Riesz theorem as follows.

Theorem 3.7 P1:[If $\mu \in M$ is autoc. and $f_n \xrightarrow{\mu} f$, then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \xrightarrow{a.e.\ \mu} f$ and $f_{n_k} \xrightarrow{p.a.e.\ \mu} f$]

P2:[If $\mu \in M$ is p.autoc. and $f_n \xrightarrow{p.\ \mu} f$ then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \xrightarrow{p.a.e.\ \mu} f$ and $f_{n_k} \xrightarrow{a.e.\ \mu} f$]

We have

Theorem 3.8 P1 and P2 are two dual propositions.

Note 3.4 Precisely speaking, the dual proposition of P1 is $\bar{P}1$:[if $\bar{\mu} \in M$ is p.autoc. and $\bar{f}_n \xrightarrow{p.\ \bar{\mu}} \bar{f}$, then there exists a subsequence $\{\bar{f}_{n_k}\}$ of $\{\bar{f}_n\}$ such that $\bar{f}_{n_k} \xrightarrow{p.a.e.\ \bar{\mu}} \bar{f}$ and $\bar{f}_{n_k} \xrightarrow{a.e.\ \bar{\mu}} \bar{f}$]. But we can regard that P2 is the dual proposition of P1. The similar things shall be met in Theorem 3.8 and 3.9.

§3.5.2 On Egoroff theorem([4, Theorem 4.2])

Under the assumptions of section 2 and by using the conclusion of 3.3, we can rewrite Egoroff theorem as follows:

Theorem 3.9 P3:[if $\mu \in M$ is autoc., then $f_n \xrightarrow{a.e.\ \mu} f \Rightarrow f_n \xrightarrow{a.u.\ \mu} f$ and $f_n \xrightarrow{p.a.u.\ \mu} f$]

and P4:[if $\mu \in M$ is p.autoc.then $f_n \xrightarrow{p.a.e.\,\mu} f \Longrightarrow f_n \xrightarrow{p.a.u.\,\mu} f$ and $f_n \xrightarrow{a.u.\,\mu} f$].

We have

Theorem 3.10 P3 and P4 are dual propositions.

§3.5.3 On Lebesgue theorem ([4,Theorem 4.3])

Under the assumptions of section 2 and the conclusions of 3.3,Lebesgue theorem can be described again by

Theorem 3.11 P5:[$f_n \xrightarrow{a.e.\,\mu} f \Longrightarrow f_n \xrightarrow{\mu} f$]

P6:[$f_n \xrightarrow{p.a.e.\,\mu} f \Longrightarrow f_n \xrightarrow{p.\,\mu} f$]

P7:[$f_n \xrightarrow{a.e.\,\mu} f$ and μ is 0-add.$\Longrightarrow f_n \xrightarrow{p.\,\mu} f$]

P8:[$f_n \xrightarrow{p.a.e.\,\mu} f$ and μ is P.0-add.$\Longrightarrow f_n \xrightarrow{\mu} f$]

We have

Theorem 3.12 the dual proposition of P5 and P7 are P6 and P8 repectively.

Theorem 3.8,theorem 3.10 and theorem 3.12 reveal the dual relations in Riesz theorem,Egoroff theorem and Lebesgue theorem.

§4. On the dual problems for fuzzy integral

§4.1 One proposition for fuzzy integral

Theorem 4.1 $\forall f \in F, \forall \mu \in M$,there holds $\int f d\mu = \mu(x) - \int \bar{f} d\bar{\mu}$

Proof:From the definition of fuzzy integral,we have

$$\int f d\mu = \bigvee_{\alpha > 0} [\alpha \wedge \mu(f > \alpha)]$$

$$= \bigvee_{0 < \alpha < \mu(X)} [\alpha \wedge \mu(f > \alpha)]$$

$$= \bigwedge_{0 < \alpha < \mu(X)} [\alpha \vee \mu(f > \alpha)] \text{ (see Kandel [8])}$$

$$= \mu(x) - \bigvee_{0 < \alpha < \mu(X)} [\mu(x) - (\alpha \vee \mu(f > \alpha))]$$

$$= \mu(x) - \bigvee_{0 < \alpha < \mu(X)} [(\mu(x) - \alpha) \wedge (\mu(x) - \mu(f > \alpha))]$$

$$= \mu(x) - \bigvee_{0 < \alpha < \mu(X)} [(\mu(x) - \alpha) \wedge \bar{\mu}(f < \alpha)]$$

$$= \mu(X) - \bigvee_{0 < \mu(x) - \alpha < \mu(x)} [(\mu(X) - \alpha) \wedge (\bar{\mu}(\bar{f} > \mu(X) - \alpha)]$$

$$= \mu(X) - \bigvee_{0<\alpha<\mu(X)} [\alpha \wedge \bar{\mu}(\bar{f} > \alpha)]$$

$$= \mu(X) - \int \bar{f} d\bar{\mu} \quad (\text{note } \bar{\mu}(X) = \mu(X))$$

§4.2 The dual proposition rules for fuzzy integral

Let P be a proposition for fuzzy integral. If we replace the fuzzy measure μ, the measurable function f and the value of the fuzzy integral "$\int$" in P by their dual ones $\bar{\mu}$, $\bar{f}$ and $\mu(x)-\int$ respectively, then we can obtain a new proposition for fuzzy integral, denote it by $\bar{P}$. We call $\bar{P}$ the dual proposition of P. From Theorem 4.1, we can know that a proposition P for fuzzy integral and its dual proposition $\bar{P}$ satisfy that P is true <==> $\bar{P}$ is true. When a proposition for fuzzy integral concerns with a sequence of measurable functions with one finds of convergence, we must replace the convergent condition by its dual one in proposition $\bar{P}$. For instance, $[f_n \xrightarrow{a.e.\ \mu} f]$ must be replaced by $[\bar{f}_n \xrightarrow{p.a.e.\ \bar{\mu}} \bar{f}]$ etc.

§4.3 Application of the dual proposition rules for fuzzy integral

§4.3.1 On the monotone convergence theorems for fuzzy integral

Theorem 4.2. The dual proposition of P: $[\forall f_n, f \in F,\ f_n \uparrow f \Longrightarrow \int f_n d\mu \uparrow \int f d\mu]$ is $\bar{P}$: $[\forall f_n, f \in F,\ f_n \downarrow f \Longrightarrow \int f_n d\mu \downarrow \int f d\mu]$.

Note 4.1. According to the dual rules for fuzzy integral, the dual proposition of P in Theorem 4.2 should be $\bar{P}'$: $[\bar{f}_n \downarrow \bar{f} \Longrightarrow \mu(x) - \int \bar{f}_n d\bar{\mu} \uparrow \mu(x) - \int \bar{f} d\bar{\mu}]$ or equivalently, $\bar{P}''$: $[\bar{f}_n \downarrow \bar{f} \Longrightarrow \int \bar{f}_n d\bar{\mu} \downarrow \int \bar{f} d\bar{\mu}]$. We can easily know that $\bar{P}$ and $\bar{P}''$ are the same conclusion since f_n, f and μ are arbitrary. In Theorem 4.3 and Theorem 4.4 we shall meet the similar cases.

§4.3.2 On the convergence almost everywhere theorem and convergence pseudo-almost everywhere theorem for fuzzy integral

Theorem 4.3 The dual proposition of P: $[\forall f_n, f \in F, f_n \xrightarrow{a.e.\ \mu} f \Longrightarrow \int f_n d\mu \to \int f d\mu$ iff μ is O-add.] is $\bar{P}$: $[\forall f_n, f \in F, f_n \xrightarrow{p.a.e.\ \mu} f \Longrightarrow \int f_n d\mu \to \int f d\mu$ iff μ is P.O-add.]

§4.3.3 On the convergence theorem in fuzzy measure and convergence theorem pseudo- in fuzzy measure for fuzzy integral

Theorem 4.3 The dual proposition of

$P:[\forall f_n, f \in F, f_n \xrightarrow{\mu} f \Longrightarrow \int f_n d\mu \to \int f d\mu$ iff μ is autoc.] is $\bar{P}:[\forall f_n, f \in F, f \xrightarrow{p.\mu} f \Longrightarrow \int f_n d\mu \to \int f d\mu$ iff μ is p.autoc.].

Theorem 4.2, theorem 4.3 and Theorem 4.4 describe the dual relations between the propositions of convergence theorems for fuzzy integral.

§5. Conclusion

In section 3 and section 4, we list some dual concepts and dual propositions in the theorem of fuzzy measure and fuzzy integral. Generally speaking, for any concept and propositions for fuzzy mesaure and fuzzy integral, we can give the corresponding dual concept and dual proposition. In other word, in the theory of fuzzy mesaure and fuzzy integral, introducing one concept is equivalent to introducing two concepts and proving one proposition is equivalent to proving two propositios. These conclusions make us understand the inner relations between some concepts and some propositions in the theory of fuzzy measure and fuzzy integral.

At last, we point out that some concepts which seem dual are not dual concepts in the case of this paper. For instance, [$\mu \in M$ is sub-additive] and [$\bar{\mu} \in M$ is super-additive] are not dual concepts.

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