FUZZY INTEGRALS OF SET-VALUED MAPPINGS AND FUZZY MAPPINGS

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The paper based on [4] first defines the concept of fuzzy integrals of set-valued mappings by the similar way of Aumann[1]. Then it is discussed, some results similar as Aumann integral are obtained, these include convexity, closedness, Fatou's lemma and Lebesgue convergence theorem, etc. At last, the fuzzy integral of fuzzy mappings is defined, the extended results corresponding with fuzzy integrals of set-valued mappings is showed.

1. Preliminaries

In the paper, the following notations and concepts will be used. Symbol I denotes the unite interval [0, 1], P(I) denotes the power set of I, $\mathcal{F}$(I) denotes the fuzzy power set of I. The triplet (X, $\mathcal{A}$ m) is a classical complete probability space (non fuzzy), $\mu:\mathcal{A}\longrightarrow I$ is a fuzzy measure of Sugeno's sense, and in addition, μ satisfies the following two conditions:

(i) μ is null-additive, i.e. $\mu(A)=0$ implies $\mu(A\cup B)=\mu(B)$;

(ii) $\mu \ll m$, i.e. $m(A)=0$ implies $\mu(A)=0$.

Throughout the paper, our dissussion is supposed to be carried on (X, $\mathcal{A}$, m, μ).

The fuzzy integral of a measurable function $f: X\longrightarrow I$ on X is defined as

$$\int f d\mu = \bigvee_{\alpha\in I} [\alpha \wedge \mu(f\geqslant \alpha)]$$

where $(f\geqslant \alpha)=\{x\in X; f(x)\geqslant \alpha\}$

Lemma. Let f_1, f_2 be two measurable functions, then $f_1(x)=f_2(x)$ for $x\in X$ m-a.e implies $\int f_1 d\mu=\int f_2 d\mu$.

A set-valued mapping is a mapping $F: X\longrightarrow P(I)\setminus\{\phi\}$, and it is said to be measurable if its gragh is measurable, i.e. $GrF=\{(x,r)\in X\times I: r\in F(x)\}$ is belongs to $\mathcal{A}\otimes Borel(I)$.

A fuzzy mapping is a mapping $\tilde{F}: X\longrightarrow \mathcal{F}(I)\setminus\{\phi\}$, and it is said to be measarable if its λ-cutting set-valued mapping F_λ is measurable for every $\lambda\in(0,1]$, where $F_\lambda(x)=[\tilde{F}(x)]_\lambda=\{r\in I: \tilde{F}(x)(r)\geqslant\lambda\}$.

2. Fuzzy integrals of set-valued mappings

Definition 2.1 Let F be a set-valued mapping, then the fuzzy integral of F on X is defined as

$\int F d\mu=\{\int f d\mu : f\in S(F)\}$

where $S(F)=\{f: f$ is measurable, $f(x)\in F(x)$ for $x\in X$ m-a.e$\}$, i.e. the family of m-a.e measurable selections of F.

Obviously, $\int F d\mu$ may be empty.

A set-valued mapping F is said to be integrable if $\int F d\mu\neq\phi$.

Proposition 2.1 If F is a measurable set-valued mapping, then F is integrable.

A set-valued mapping F is said to be convex-valued, if F(x) is convex for all $x\in X$ m-a.e.

Propsition 2.2 If a measarble set-valued mapping F is convex-valued, then $\int F d\mu$ is convex.

For a set-valued mapping F, let coF(x) denote the convex hull of F(x) for $x\in X$ m-a.e, then holds the following.

Proposition 2.3 If F is a measurable set-valued mapping, then $co\int F d\mu=\int coF d\mu$.

A set-valued function F is said to be closed-valued if F(x) is closed for $x\in X$ m-a.e.

Proposition 2.4 If a measurable set-valued function F is closed-valued, then $\int F d\mu$ is closed.

Corollary 1. Let F be a measurable interval-valued function, i. e. $F(x)=[f^-(x), f^+(x)]$, then $\int F d\mu=[\int f^- d\mu, \int f^+ d\mu]$

Let $\{A_n\}\subset P(I)$ be a sequence, we define

$\text{Limsup } A_n=\{x; x=\lim_{k\to\infty} x_{n_k}, x_n\in A_n(n\geqslant 1)\}$

$\text{Liminf } A_n=\{x; x=\lim x_n, x_n\in A_n(n\geqslant 1)\}$

It had been shown that Limsup A_n and Liminf A_n are closed sets[5]. If Limsup A_n=LiminfA_n=A, then we say $\{A_n\}$ is convergent to A, simply writed by Lim $A_n=A$ or $A_n\to A$. Using above definition, let $\{F_n\}$ be a sequence of set-valued mappings, we can define Limsup F_n, Liminf F_n, Lim F_n by pointwise way.

Theorem 2. 1 (Fatou's lemma) If $\{F_n\}$ is a sequence of measurable set-valued mappings, then

(i) $\text{Limsup} \int F_n d\mu \subset \int \text{Limsup } F_n d\mu$

(ii) $\int \text{Liminf } F_n\, d\mu \subset \text{Liminf} \int F_n d\mu$

Theorem 2.2 Let $\{F_n\}$ be a sequence of measurable set-valued mappings, F be a set-valued mapping. If Lim $F_n=F$, then $\text{Lim} \int F_n d\mu=\int F d\mu$.

3. Fuzzy integrals of fuzzy mappings

Definition 3.1 Let $\tilde{F}$ be a fuzzy mapping. Then the fuzzy integral of $\tilde{F}$ on X is defined as

$(\int \tilde{F} d\mu)(r)=\sup\{\lambda\in(0,1]; r\in \int F_\lambda d\mu\}$

A fuzzy set $\tilde{r}\in \mathcal{F}(I)$ is said to be fuzzy convex (fuzzy closed) if $r_\lambda=\{r\in[0,1]; \tilde{r}(r)\geqslant\lambda\}$ is convex (closed) for each $\lambda\in(0,1]$. A fuzzy mapping $\tilde{F}$ is said to be fuzzy convex-valued (fuzzy closed-valued) if $\tilde{F}(x)$ is fuzzy convex (fuzzy closed) for $x\in X$ m-a. e.

Proposition 3.1 If $\tilde{F}$ is a measurable fuzzy mapping, then

(i) $\tilde{F}$ is fuzzy convex-valued implies $\int \tilde{F} d\mu$ is fuzzy convex;

(ii) $\tilde{F}$ is fuzzy closed-valued implies $\int \tilde{F} d\mu$ is fuzzy closed, and $(\int \tilde{F} d\mu)_\lambda=\int F_\lambda d\mu$ for $\lambda\in(0,1]$.

Corollary 2. If a measurable fuzzy mapping $\tilde{F}$ is a fuzzy number mapping, then $\int \tilde{F} d\mu$ is a fuzzy number.

Let $\{\tilde{\Gamma}_n\}$ be a sequence of fuzzy sets. Its limit superior and limit inferior are defined as

$$(\text{Limsup } \tilde{\Gamma}_n)(r)=\{\lambda\in(0,1]; r\in \text{Limsup}(\tilde{\Gamma}_n)_\lambda\}$$

$$(\text{Liminf } \tilde{\Gamma}_n)(r)=\{\lambda\in(0,1]; r\in \text{Liminf}(\tilde{\Gamma}_n)_\lambda\}$$

If Limsup $\tilde{\Gamma}_n$=Liminf $\tilde{\Gamma}_n=\tilde{\Gamma}$, then we say that $\{\tilde{\Gamma}_n\}$ is convergent to $\tilde{\Gamma}$, simply writed by Lim $\tilde{\Gamma}_n=\tilde{\Gamma}$ or $\tilde{\Gamma}_n\to\tilde{\Gamma}$.

Similarly, for a sequence of fuzzy mappings $\{F_n\}$, Limsup F_n, Liminf F_n, and Lim F_n are just defined with pointwise (m-a.e).

Theorem 3.1 (Fatou's lemma) If $\{\tilde{F}_n\}$ is a sequence of measurable fuzzy mappings, then

(i) Limsup $\int\tilde{F}_n d\mu \,\tilde{\subset}\, \int$Limsup $\tilde{F}_n d\mu$

(ii) $\int$Liminf $\tilde{F}_n d\mu \,\tilde{\subset}\,$ Liminf $\int\tilde{F}_n d\mu$.

Theorem 3.2 (Lebesegue convegence theorem) If $\{\tilde{F}_n\}$ is a sequence of measurable fuzzy mappings, then $\{\tilde{F}_n\}$ is convergent implies $\int\tilde{F}_n\, d\mu$ is convergent, and $\int \text{Lim}\tilde{F}_n\, d\mu=\text{Lim}\int\tilde{F}_n d\mu$.

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