FUZZY SYSTEMS WITH DEFUZZIFICATION: THE EQUIVALENT CRISP SYSTEMS

Horia-Nicolai Teodorescu
Center for A.I. & Fuzzy Systems
Polytechnic Institute of Iasi
Iasi 6600, Romania

Key words: fuzzy systems; nonlinear systems; approximation; interval analysis; triangular membership function; center of gravity defuzzifier; staircase characteristic.

Abstract: A fuzzy system with defuzzified output represents a crisp system build in a specific way, similar to the way the expert systems are obtained. A methodology is derived to build up a fuzzy system -- with center-of-gravity defuzzification -- that approximates a given crisp system. It is proved that almost any crisp system can be approximated by a fuzzy one, with simple membership functions for the input and the output.

1. INTRODUCTION

Consider a logic fuzzy system with real-valued inputs (i.e. fuzzified input) and real-valued output(s). The output of the fuzzy system is defuzzified. In this paper the defuzzifier is supposed to operate by the center of gravity method. From the user point of view, a fuzzy system with defuzzified output is equivalent to a crisp system. Following, for the designer of a crisp system based on a defuzzified logic fuzzy system, several natural problems have to be clearly solved [2], namely:

i) being given a fuzzy system (as above), find the crisp system (i.e. its characteristic function) equivalent to it;

ii) conversely, being given a crisp system, build up a fuzzy system equivalent to it -- if this is possible, i.e.:

iii) are there crisp systems which characteristic function can not be obtain by using a fuzzy system?, or, equivalently: are fuzzy systems a real generalization of crisp systems?

iv) if it is difficult (or impossible) to build up a fuzzy system which characteristic function is the same of that of a given crisp system, is it possible to approximate with as low as needed error a crisp system by a fuzzy system?

These problems are of obvious practical interest in fuzzy systems applications, including fuzzy control. Thus, a series of papers is devoted to the topic.

Part of these problems were solved in [1], [2]. In this paper, we remember a few basic results and we generalize some propositions.

2. THE CHARACTERISTIC FUNCTIONS OF A CLASS OF FUZZY SYSTEMS

2.1. Fuzzy systems with interval-valued inputs and outputs

In this paper, only one-input, one-output fuzzy systems are considered.

This section is aimed to exemplify the way the results are got. Although the analysis is simple to perform, it is rather cumbersome for the general case. Following, in this section, only a very simple example is dealt with. The extension to the general case appears to be elementary.

Let us consider a strictly monotonic system, i.e. a system described by the set of rules:

R_k: IF x is A_k THEN y is B_h

R_{k+1} IF x is A_{k+1} THEN y is B_{h+1}

$\forall$ k = 1, 2,...,n-1. Above, x stands for the input value, y for the output value, A_k are the linguistic degrees of the input, B_h are the l.d. of the output, R_k denotes the k-th rule, k = 1,... , n ; h = 1,..., m; (m > n), and A_k and B_h are ranged in the natural increasing order. (Note that above we considered a 'strictly increasing' system. By taking in R_{k+1} the linguistic degrees of y to be B_{h-1}, one gets the strictly decreasing system).

Consider that the membership functions (m.f.) of A_k and B_h are interval-type m.f., i.e.:

$$\mu_A : X \dashrightarrow [0,1] \; ; \; a^{(1)}_k \le a^{(1)}_{k+1} \le a^{(2)}_k.$$

$$\mu_{Ak}(u) = \begin{cases} 0 & \text{for: } u < a^1_k \\ 1 & a^{(1)}_k \le u \le a^{(2)}_k \\ 0 & u < a^{(2)}_k \end{cases}$$

and the membership functions $\mu_{B_h}(v)$: Y --> [0,1] are given by:

$$\mu(v) = \begin{cases} 0 & v < b^{(1)}_h \\ 1 & b^{(1)}_h \le v \le b^{(2)}_h \\ 0 & v < b^{(2)}_h \end{cases}$$

The input-to-output mapping for such a system is sketched in Figure 1.

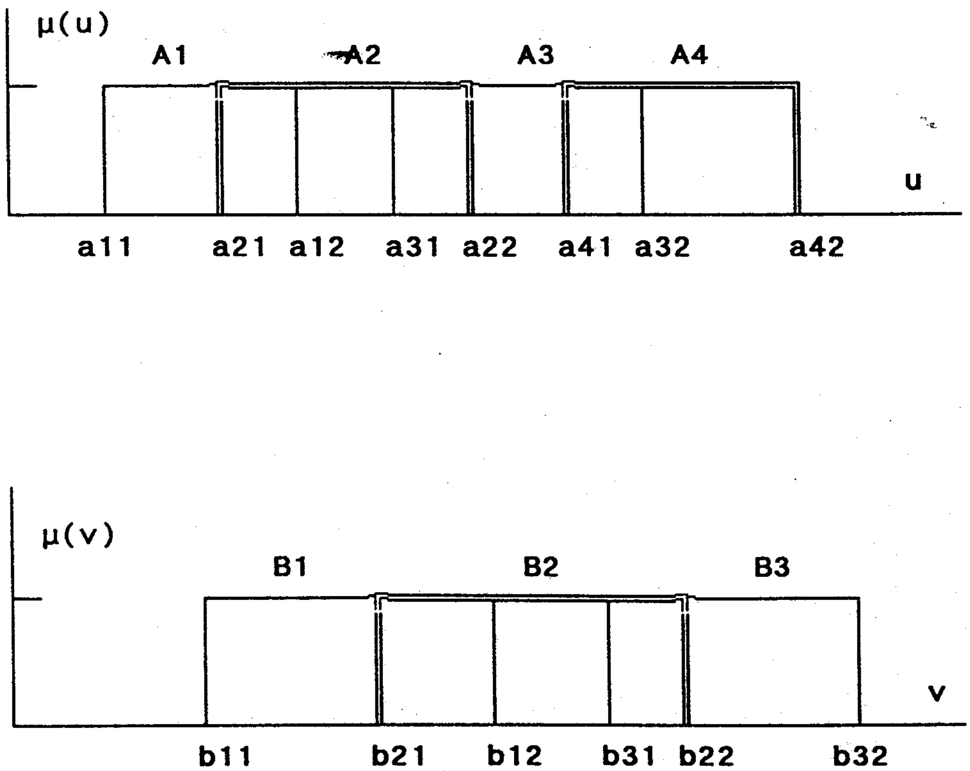

Figure 1: Input to output mapping with interval-valued m.f.

We shall suppose the following natural conditions are satisfied: the input (and output) intervals overlap only in pairs, i.e.:

$$A_k \cap A_{k+1} \cap A_{k+2} = \emptyset \; ; \qquad B_h \cap B_{h+1} \cap B_{h+2} = \emptyset$$

$$\forall\, k = 1,\ldots,n-2; \qquad \forall\, h = 1,\ldots,m-2.$$

(These conditions are generally satisfied in usual applications).

The following simple statement holds:

Proposition 1. [1], [2] The characteristic functions of the above described systems are staircase increasing functions given by:

$$y\,(x) \;=\; < \begin{cases} b^{(2)}_{h+1} - b^{(1)}_{h})/2 & a^{(1)}_{k+1} < a^{(2)}_{k} \\ (b^{(2)}_{h} - b^{(1)}_{h})/2 & a^{(2)}_{k-1} < a^{(1)}_{k+1} \\ (b^{(2)}_{h} - b^{(1)}_{h-1})/2 & a^{(1)}_{k} < a^{(2)}_{k-1} \end{cases}$$

Proof:It is easy to see that for the first interval of x in the above equation, the output is $B_h \cup B_{h+1}$; thus, the center of gravity is given by the average of the limits of the composed interval. In the same manner, one determines the output for the other two cases. Not included in this discussion (and in the above description of the output) are the cases

$$a^{(1)}_{1} \le x \le a^{(1)}_{2}\ , \text{ and}$$

$$a^{(2)}_{n-1} < x \le a^{(2)}_{n},$$

which are solved in the same manner.

The characteristic function of the fuzzy system is plotted in Figure 2.

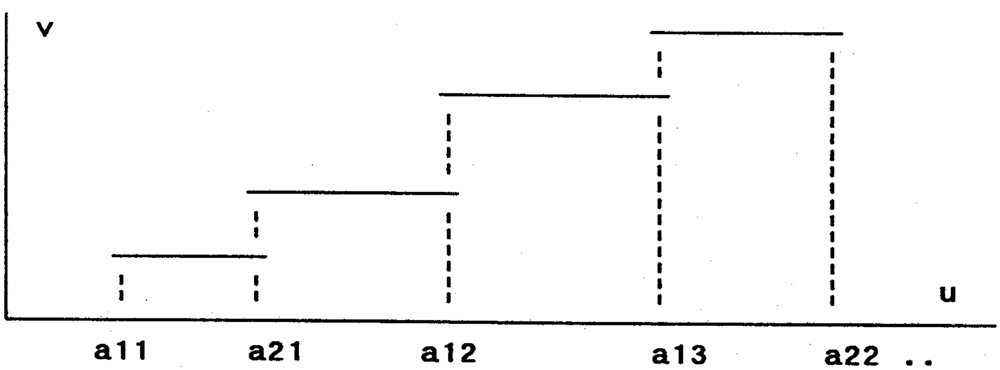

Figure 2: Characteristic function of the systems

In a similar manner can be dealt with the case of multiple outputs/inputs fuzzy systems.

The following elementary properties of the characteristic functions of the discussed fuzzy systems can be useful in applications:

i) The number of jumps of the characteristic function is 2n - 1, where n is the number of linguistic degrees of the input.

ii) The jumps are equally distributed if the points $a^{1,2}_{k}$ are uniformly distributed.

iii) The jumps are equal if the points $b^{1,2}_{h}$ are equally distanced; then, the jumps are equal to the distance between two subsequent points.

Now remember that any real and real-valued function which is almost continuous (in the sense that it has a numerable set of discontinuity points) can be approximated by a staircase function. (In the strict sense, a staircase function is monotonic; we use the term to denote any function constant on a numerable set of intervals which are disjoint and cover the real axis). Using this basic result, one gets the following useful result:

Proposition 2. [1], [2] Any nonlinear crisp system having a monotonic characteristic function can be approximated by a fuzzy system having interval-valued inputs and outputs.

Proposition 1 indicates the class of fuzzy systems that are equivalent to crisp systems from a specific class: the staircase type characteristic function crisp systems. Proposition 2 allows to approximate a very large class of crisp systems by means of fuzzy systems of the simplest type -- i.e. using just interval valued inputs and outputs.

2.2. Fuzzy systems with interval-valued inputs and any output membership functions

It is easy to check the following property:

Proposition 3: Let be a logic n-inputs, m-outputs fuzzy system. Let us denote by A_{hi} the h-th input sets, and by B_{kj} the k-th output sets. Let the system be described by a fixed set of rules:

IF A_{1i1} & ... & A_{nin} THEN B_{1k1} & ... & B_{1m1}

The inputs to the system are supposed to be interval-valued. The outputs of the system are defuzzified by the center of gravity method. Then:

i) The characteristic function of the defuzzified system is a staircase function for any shape of the membership functions of the outputs B_{kj} (see Figure 3).

ii) Suppose that the membership functions of the output sets of the fuzzy system have no fixed shapes, but the interval on which they do not vanish are fixed (i.e. the membership functions of the various sets B_{kj} chosen to represent the same output does not vanish on the same interval). Then, the number of stairs is fixed and the intervals defined by the stairs are not dependent on the output membership functions.

iii) If condition ii) is fulfilled and the output membership functions of the sets chose to represent B_{kj} have the same center of gravity, then the change in the shape of the output membership functions does not change the characteristic function of the system with defuzzified output(s).

The proof is elementary and follows the ideas in the proof of Proposition 1.

The above result slightly generalizes a proposition proved in [2].

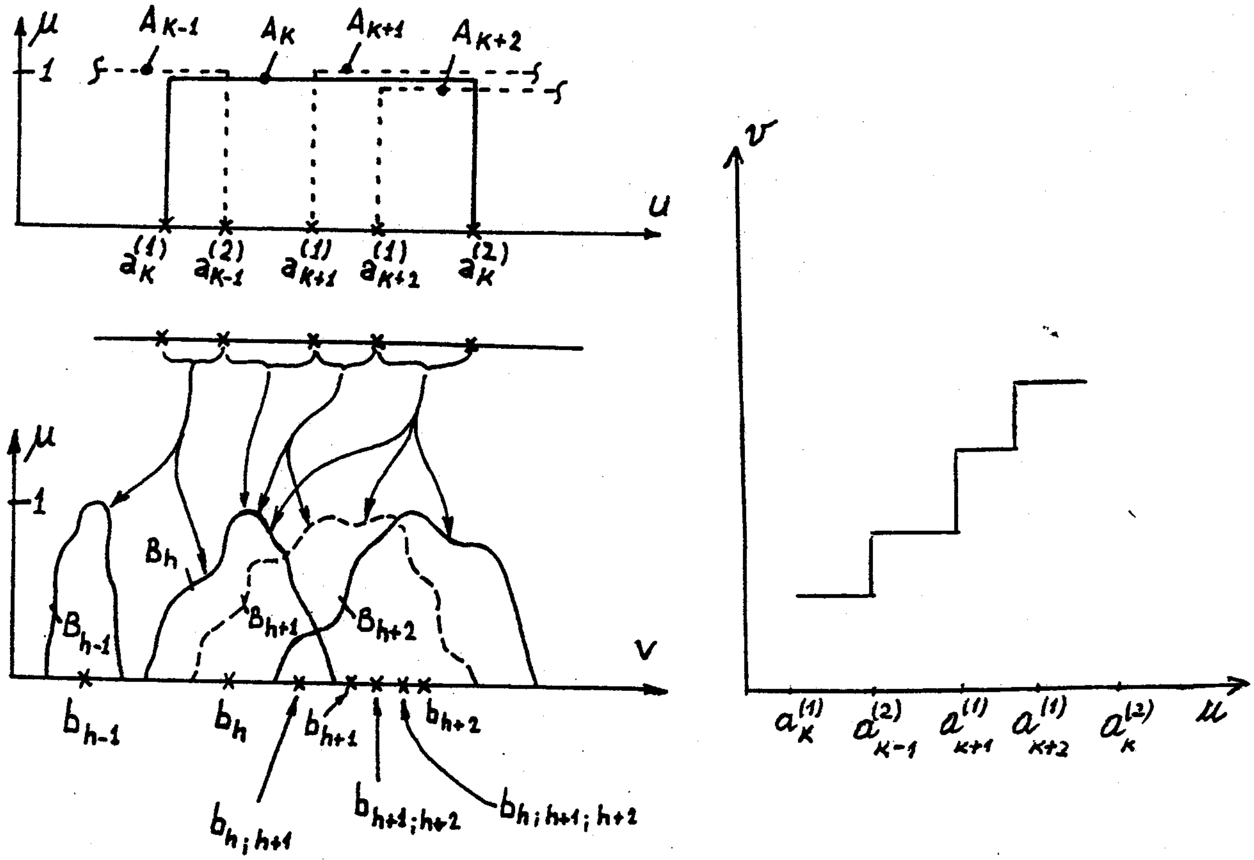

Figure 3: The fuzzy system according to Proposition 3 and the characteristic function of the corresponding crisp system

By similar reasoning, one gets:

THEOREM: Any nonlinear crisp system having continuous characteristic function(s) can be approximated by a fuzzy system with interval-valued membership functions.

The proof is elementary and is based on the above propositions and on Weierstrass theorem.

Obviously, the result is easy to extend to systems with a numerable set of discontinuities of the characteristic functions.

3. CONCLUSIONS

Some conclusions are worth to derive.for applications:

i) It could be of low interest to build defuzzified - output(s) fuzzy systems with cumbersome input or output membership functions, because similar results can be obtained using an interval-valued input/output fuzzy system.

ii) The use of more complex input/output membership functions could be useful only to get more intuitive rules describing the system, or to minimize the rules number (for a fixed approximation error).

iii) The interest in using defuzzified fuzzy systems instead of classical crisp systems resides in the possibility to use the expert knowledge to build up the system, and also in its robustness.

iv) The fuzzy systems are a true generalization of the class of nonlinear systems, because any crisp system can be seen as the limit of a string of defuzzified fuzzy systems.

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