L - MODEL OF ECONOMIC DYNAMICS

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1. Introduction

Continuing our considerations on L - multifunctions and their applications we are using the abbreviations and notations from [1] and [2]. In this paper we introduce the concept of L - economic model. Moreover the concepts of technological trajectories of such a model and their optimality are here presented.

2. Desription of a model.

Let E be a subset of R with at least two different elements. It is assumed that $0 \in E$. Elements of E will be called time moments and the element 0 initial time moment.
Let $\tilde{E} = \{(t,\tau) \in E \times E : t<\tau\}$.

Definition 1. An L - model of economic dynamics is an object

$$M = \{E, (R^{n_t})_{t\in E}, (K_t)_{t\in E}, (F_{t,\tau})_{(t,\tau)\in\tilde{E}}\},$$

where

- R^{n_t} denotes n_t - dimensional Euclidean space,
- $K_t = R_+^{n_t}$,
- $F_{t\tau} : K_t \rightsquigarrow P(K_\tau)$ - an L - multifunction satisfying the following properties:
  - for any $(t,\tau) \in \tilde{E}$ $F_{t\tau}$ is
    1. superadditive , [1],
    2. conical , [1],
    3. closed , [2],
    4. $(0,y,r,t) \notin W_{F_{t\tau}}$ for $y \neq 0$
  - if $t,\tau,\theta \in E$ and $t<\theta<\tau$ then

$F_{t\tau} = F_{\theta\tau} \circ F_{t\theta}$, [1].

In the classical model the technology of an economy in a time interval (e.g.from moment t to moment τ)is described by a multifunction . In this way,a certain set of goods is assigned to a certain good. Now, we will assume that we have at our disposal information about the qualities of the outlays goods at moment t. As a result of applying a technology which defines the economy in a time interval $\langle t,\tau\rangle$, a set of goods of given qualities can be abtained from good x_t of e.g. r_t quality. Of course, the qualities of abtained goods depend on the quality of the outlays goods and the technology. However, it may happen that the application of the same technology to the same good yields goods of different qualities. The qualities will be described by numbers of a unit interval.

From the above line of reasoning it foollows that in a time interval $\langle t,\tau\rangle$ we have the following information at our disposal:

- a goods which are the outlays at moment t,
- quality of the outlays good,
- a transformation assigning an outlays goods to a set of goods of given qualities.

The mentioned transformation we will described by the L - multifunction.

D e f i n i t i o n 2. A technological trajectory of M is a family $Tr = (\{x_t,r_t\})_{t\in E}$ such that

(a) $x_o \in K_o$,
(b) $r_t \neq 0$ for any $t \in E$,
(c) $(x_t,x_\tau,r_t,r_\tau) \in W_{F_{t\tau}}$ for any $(t,\tau) \in \tilde{E}$.

If $Tr = (\{x_t,r_t\})_{t\in E}$ is a technological trajectory of M,then the fuzzy singleton $\{x_t,r_t\}$ is called the state of the trajectory Tr at the time t; $\{x_o,r_o\}$ is the initial state of Tr. It is said that the fuzzy trajectory Tr goes out $\{x,r\}$ if $\{x,r\} = \{x_o,r_o\}$ and passes through $\{x,r\}$ at the time moment t if the state of Tr at the time moment t is $\{x,r\}$.

It can be seen that in the case of a technological trajectory of M it is possible to determine at each stage the degree of quality r_t of the abtained good x_t.

Let $W_{t\tau}$ denote the graph of an L - multifunction $F_{t\tau}$. Now, we will formulate a theorem of the existence of a technological trajectory in our model.

Theorem 1. (Existence of technological trajectory) Let $(0,\tilde{t}) \in \tilde{E}$ and $(y_0, y_{\tilde{t}}, r_0, r_{\tilde{t}}) \in W_{0\tilde{t}}$, $r_0 \neq 0$, $r_{\tilde{t}} \neq 0$. Then there exists a technological trajectory Tr of M going out $\{y_0, r_0\}$ and passing through $\{y_{\tilde{t}}, r_{\tilde{t}}\}$ at the time moment $\tilde{t}$.

Proof. For every $t \in E$ let us take element $u_t \in R^{n_t}$ such that $u_t \notin K_t$ and let us denote $L_t = (K_t \cup \{u_t\}) \times I$. Let $L = \prod_{t \in E} L_t$. Next let us choose a subset $S \subset L$ of such elements $Tr = (\{x_t, r_t\})_{t \in E}$ that there exists a subset $E_{Tr} \subset E$ such that

(a) $0, \tilde{t} \in E_{Tr}$,

(b) if $t, \tau \in E_{Tr}$ and $t < \tau$ then $(x_t, x_\tau, r_t, r_\tau) \in W_{t\tau}$,

(c) $\{x_0, r_0\} = \{y_0, r_0\}$, $\{x_{\tilde{t}}, r_{\tilde{t}}\} = \{y_{\tilde{t}}, r_{\tilde{t}}\}$,

(d) if $t \in E \setminus E_{Tr}$ then $x_t = u_t$.

By the way, let us mention that $S \neq \emptyset$. Indeed, the element $Tr = (\{x_t, r_t\})_{t \in E}$ such that
$\{x_0, r_0\} = \{y_0, r_0\}$, $\{x_{\tilde{t}}, r_{\tilde{t}}\} = \{y_{\tilde{t}}, r_{\tilde{t}}\}$, $\{x_t, r_t\} = \{u_t, r_t\}$
if $t \in E$ and $t \neq 0, \tilde{t}$ belongs to S and $E_{Tr} = \{0, \tilde{t}\}$.

In S one can introduce a partial ordering $\succ$ as follows:

$$Tr^1 \succ Tr^2, \quad Tr^i = (\{x_t^i, r_t^i\})_{t \in E} \in S, \quad i = 1,2,$$

iff

(a) $E_{Tr^1} \supset E_{Tr^2}$,

(b) $\{x_t^1, r_t^1\} = \{x_t^2, r_t^2\}, \quad \forall\, t \in E_{Tr^2}$.

Now, it will be checked that each chain in S is bounded from above. Let $(Tr^\alpha)_{\alpha \in A}$ denote a chain in S, $Tr^\alpha = (\{x_t^\alpha, r_t^\alpha\})_{t \in E}$, $\alpha \in A$. It is seen that the element $Tr = (\{x_t, r_t\})_{t \in E}$ such that

(a) $\{x_t, r_t\} = \{x_t^\alpha, r_t^\alpha\}$ of $t \in E_{Tr^\alpha}$,

(b) $\{x_t, r_t\} = \{u_t, r_t\}$ if $t \notin \bigcup_{\alpha \in A} E_{Tr^\alpha}$, belongs to S and $E_{Tr} = \bigcup_{\alpha\in A} E_{Tr^\alpha}$. Moreover, for every $\alpha \in A$ $Tr \wr Tr^\alpha$

So, the chain $(Tr^\alpha)_{\alpha\in A}$ is bounded from above, as asserted. By Zorn's Lemma the set S has a maximal element.

It remains now to prove that for each maximal element $Tr = (\{x_t, r_t\})_{t\in E}$ of S there holds $E_{Tr} = E$. Indeed, let us suppose that there exist a maximal element $Tr \in S$ such that $E \setminus E_{Tr} \neq \emptyset$. So, there exists an element $\theta \in E - E_{Tr}$. Now, let us define

$$F_1 = \{t \in E_{Tr} : t<\theta\},\ F_2 = \{t \in E_{Tr} : \theta<\tau\}, F = F_1 \times F_2,$$

and for $(t,\tau) \in F$ and $\{x_t, r_t\}, \{x_\tau, r_\tau\} \in Tr$

$$b_{t\tau} = \{\{x_\theta, r_\theta\} : (x_t, x_\theta, r_t, r_\theta) \in W_{t\theta} \text{ and } (x_\theta, x_\tau, r_\theta, r_\tau) \in W_{\theta\tau}\}.$$

(1) The subset $b_{t\tau}$ is a non - empty subset. In fact, let $t<\theta<\tau$. Because of $\{x_t, r_t\}, \{x_\tau, r_\tau\} \in Tr$ we observe that

$$(x_t, x_\tau, r_t, r_\tau) \in W_{t,\tau}$$

Next, taking into account the properties of composition of L-multifunctions, we get

$$F_{t\tau} = F_{\theta\tau} \circ F_{t\theta}.$$

This means that there exists $\{x_\theta, r_\theta\}$ such that

$$(x_t, x_\theta, r_t, r_\theta) \in W_{t\theta} \text{ and } (x_\theta, x_\tau, r_\theta, r_\tau) \in W_{\theta\tau},$$

i.e. $b_{t\tau} \neq \emptyset$.

(2) The set $b_{t\tau}$ is a compact set.

The L-multifunctions $F_{t\theta}$ and $F_{\theta\tau}$ are closed and $(0,y,r,t) \notin W_{t\theta}$, $(0,y,r,t) \notin W_{\theta\tau}$ for $y \neq 0$. Therefore $F_{t\theta}$ and $F_{\theta\tau}$ are sequentially bounded (see [2]). So, the set $b_{t\tau}$ is an intersection of closed and compact set. Therefore $b_{t\tau}$ is a compact set.

(3) If (s,τ), $(t,\tau) \in F$, $s<t$, with $\{x_s, r_s\}, \{x_t, r_t\} \in Tr$ then $b_{s\tau} \supset b_{t\tau}$.

(4) If $(t,s), (t,\tau), s<\tau$, with $\{x_s, r_s\}, \{x_\tau, r_\tau\} \in Tr$, then $b_{ts} \subset b_{t\tau}$.

The above inclusions follow from the properties of composition of L-multifunctions.

From (1), (3) and (4) it follows that the family $(b_{t\tau})_{(t,\tau)\in F}$ is centered. Compactness of $b_{t\tau}$ and centerness of the family $(b_{t\tau})_{(t,\tau)\in F}$ yields $\bigcap_{(t,\tau)\in F} b_{t\tau} \neq \emptyset$. Let $\{z_\theta, r_\theta\} \in \bigcap_{(t,\tau)\in F} b_{t\tau}$

and let us take into account an element $\overline{Tr} = (\{\bar{x}_t, \bar{r}_t\})_{t\in E}$ such that

$$\{\bar{x}_t, \bar{r}_t\} = \begin{cases} \{x_t, r_t\} \text{ if } t \in E_{Tr}, \\ \{z_\theta, r_\theta\} \text{ if } t = \theta \\ \{u_t, r_t\} \text{ if } t \in E \setminus (E_{Tr} \cup \{\theta\}) \end{cases}$$

For this element we have $E_{\overline{Tr}} = E_{Tr} \cup \{\theta\}$. So,we get $\overline{Tr} \geq Tr$ and $\overline{Tr} \neq Tr$ in contradiction with the supposed fact that Tr is maximal element in S.Therefore,there exists a technological trajectory of M with initial state $\{y_0, r_0\}$ and passing through $\{y_{\bar{t}}, r_{\bar{t}}\}$ at the time moment $\bar{t}$.

3. Optimal technological trajectories.

Now,let us additionally assume that there exists $T \in E$ such that $t \leq T$ for all $t \in E$.

D e f i n i t i o n 3.A technological trajectory Tr with initial and terminal states $\{x_0, r_0\}$ and $\{x_\tau, r_\tau\}$ respectively is called optimal if there exists a non-zero functional $p \in K_T^*$ such that

$$p(x_T) = \max_{x \in B} p(x) > 0 , \qquad (*)$$

where $B = \{x \in K_T : (x_o, x, r_o, r_T) \in W_{oT}\}$.

An element x of B is called the limiting point from above if $a\,x \notin B$ for $a>1$. For a normal covering of B we use the symbol nB.

T h e o r e m 2. A technological trajectory Tr with initial state $\{x_0, r_0\}$ and terminal state $\{x_T, r_T\}$ is optimal iff the element x_T is a limiting point from above of the set nB.

*P r o o f.*Let Tr denote optimal technological trajectory with initial and terminal states $\{x_0, r_0\}$ and $\{x_T, r_T\}$ respectively.We will prove that x_T is a limiting point from above of nB. In contrary, suppose there exists an $a>1$ such that $ax_T \in nB$. Because $p(x_T) > 0$ we get

$$p(x_T) = \max_{x \in nB} p(x) = \max p(x) \geq p(ax_T) = a\, p(x_T) > 0, \; x \in B$$

i.e. $1 \geq a$ in contradiction with $a > 1$.
Now, let us assume that x_T is a limiting point from above nB. Let S denote the sphere nB - nB and $\|\ \|_{nB}$ Minkowski's norm. It is known that this norm is monotonous and

$$nB = \{z : \|z\|_{nB} \leq 1\}.$$

Because $x_T \in B$ is a limiting point from above of nB, there holds

$$\|x_T\|_{nB} = 1.$$

Therefore, there exists a functional $p \in K_T^*$ such that

$$p(x_T) = \|x_T\|_{nB} = 1, \; \|p\| = 1.$$

So, it is proved that for the technological trajectory Tr the condition (*) is fulfilled and thas the proof is finished.

References

[1] M.Matłoka, Introduction and general properties of L-multifunction, BUSEFAL (in print).
[2] M.Matłoka, Some topological properties of L-multifunctions, BUSEFAL (in print).