ROUGH SETS VIA FUZZY SETS.

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Correspondances one to one between families of rough and fuzzy sets are here showed($\simeq$ value 1 to 1).

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Notations, recalls and references in [1] .

1. Introduction.

When we try a definition of rough operators, we dont meet a natural formulation, because the (crisp) set-theoretic operations between sure parts and possible parts do not give in general case (to see: sect. B.2 in [1])

$\underline{A} \cup \underline{B} = \underline{A \cup B}$ and $\overline{A \cap B} = \overline{A} \cap \overline{B}$, but

$\underline{A} \cup \underline{B} \subseteq \underline{A \cup B} \subseteq A \cup B \subseteq \overline{A} \cup \overline{B} = \overline{A \cup B}$ and

$\underline{A} \cap \underline{B} = \underline{A \cap B} \subseteq A \cap B \subseteq \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

The approximable support plays a basic role when we want to define the rough operators.

Now we are going to examine the main proprieties for a good definition.

Let Ω be an universe of discourse, π a partition of Ω , $\ddot{P} = (\underline{P}, \overline{P})$ (with $\underline{P} \subseteq \overline{P}$, $\underline{P}, \overline{P} \subseteq \pi$) a (Pawlak's abstract) rough set and $\ddot{\mathcal{P}} = \ddot{\mathcal{P}}_{\pi}(\Omega)$ (abstract) pawlakean of Ω sub π (totality of rough sets).

If $\ddot{A},\ddot{B}\in\ddot{\mathcal{P}}$ let be:

$$\ddot{\mathcal{U}}_{\ddot{A},\ddot{B}} = \{\ddot{P}=(\underline{P},\overline{P})\in\ddot{\mathcal{P}} \mid \underline{P}\supseteq\underline{A}\cup\underline{B},\ \overline{P}=\overline{A}\cup\overline{B}\},$$

$$\ddot{\mathcal{U}} = \{\ddot{\mathcal{U}}_{\ddot{A},\ddot{B}}\}_{(\ddot{A},\ddot{B})\in\ddot{\mathcal{P}}\times\ddot{\mathcal{P}}};$$

$$\ddot{\mathcal{V}}_{\ddot{A},\ddot{B}} = \{\ddot{P}=(\underline{P},\overline{P})\in\ddot{\mathcal{P}} \mid \underline{P}=\underline{A}\cap\underline{B},\ \overline{P}\subseteq\overline{A}\cap\overline{B}\},$$

$$\ddot{\mathcal{V}} = \{\ddot{\mathcal{V}}_{\ddot{A},\ddot{B}}\}_{(\ddot{A},\ddot{B})\in\ddot{\mathcal{P}}\times\ddot{\mathcal{P}}};$$

$$\ddot{\mathcal{C}}_{\ddot{A}} = \{\ddot{P}=(\underline{P},\overline{P})\in\ddot{\mathcal{P}} \mid \underline{P}=\Omega-\overline{A},\ \overline{P}=\Omega-\underline{A}\}\ [= (\Omega-\overline{A},\Omega-\underline{A})],$$

$$\ddot{\mathcal{C}} = \{\ddot{\mathcal{C}}_{\ddot{A}}\}_{\ddot{A}\in\ddot{\mathcal{P}}}\ [\equiv \ddot{\mathcal{P}}]$$

If $\ddot{\cup}, \ddot{\cap}, \ddot{\complement}$ are the rough abstract union, r.a. intersection, r.a. complementation, resp., for a good definition must be:

$\ddot{A}\,\ddot{\cup}\,\ddot{B}\in\ddot{\mathcal{U}}_{\ddot{A},\ddot{B}}$, $\ddot{A}\,\ddot{\cap}\,\ddot{B}\in\ddot{\mathcal{V}}_{\ddot{A},\ddot{B}}$, $\ddot{\complement}\ddot{A}\in\ddot{\mathcal{C}}_{\ddot{A}}$ and (DeMorgan's Laws)

$$\ddot{\complement}(\ddot{A}\,\ddot{\cup}\,\ddot{B}) = \ddot{\complement}\ddot{A}\,\ddot{\cap}\,\ddot{\complement}\ddot{B},\quad \ddot{\complement}(\ddot{A}\,\ddot{\cap}\,\ddot{B}) = \ddot{\complement}\ddot{A}\,\ddot{\cup}\,\ddot{\complement}\ddot{B}.$$

2. Fuzzy supports and cores.

Let be: $\breve{\mathcal{B}} = \breve{\mathcal{Z}}^{0}_{\Lambda}$ the Λ-fuzzy boolean of Ω (to see: sect.A.1 in [1]) (totality of cantorian fuzzy sets of Ω sub Λ).

Definition.

$S_{\breve{A}} = \bigcap_{\substack{\breve{B}\in\breve{\mathcal{B}}\\ \breve{B}\supseteq\breve{A}}}$ fuzzy cantorian support of $\breve{A}$,

$K_{\breve{A}} = \bigcup_{\substack{\breve{B}\in\breve{\mathcal{B}}\\ \breve{B}\subseteq\breve{A}}}$ fuzzy cantorian core (or kernel) of $\breve{A}$.

Proposition.

$\breve{B}\in\breve{\mathcal{B}} \Rightarrow K_{\breve{B}} = S_{\breve{B}} = \breve{B}$; $\breve{A}\notin\breve{\mathcal{B}} \Rightarrow K_{\breve{A}}\subset\breve{A}\subset S_{\breve{A}}$; $K_{\breve{A}}\subsetneq S_{\breve{A}}$.

In (crisp totality of fuzzy sets) $\breve{\mathcal{Z}} = \breve{\mathcal{Z}}_{\Lambda}(\Omega)$ we crisply define the next relation $\mathcal{J}$:

$\breve{E}\,\mathcal{J}\,\breve{F}$ $(\breve{E},\breve{F}\in\breve{\mathcal{Z}}) \Leftrightarrow S_{\breve{E}} = S_{\breve{F}}$ and $K_{\breve{E}} = K_{\breve{F}}$.

Trivially, $\mathcal{J}$ is an equivalence relation and it's possible to (crisply) define the quotient set $q = \tilde{\mathcal{Z}}/_{\mathcal{J}}$: its elements represents families of fuzzy sets with same fuzzy support and core. It results $q \simeq \tilde{\mathcal{Z}}_{\{0,\xi,1\}}(\Omega)$, where $\{0,\xi,1\}$ is a lattice with 3 elements ($0 \prec \xi \prec 1$).
Moreover, if $\vartheta : q \to]0,1[$ is (under choice axiom) an application (by writing: $\vartheta([\tilde{F}]) = \xi_{[\tilde{F}]}$), we considere the (crisp) subfamily $\{\tilde{F}_\vartheta\}_{\tilde{F} \in \tilde{\mathcal{Z}}}$ such that:

$$\tilde{F}_\vartheta(x) = \begin{cases} 1 & \text{if } K_{\tilde{F}}(x) = 1 \\ 0 & \text{if } S_{\tilde{F}}(x) = 0 \\ \xi_{[\tilde{F}]} & \text{if } K_{\tilde{F}}(x) \neq S_{\tilde{F}}(x) \end{cases}$$

Then, by $[\tilde{F}_\vartheta]_{\mathcal{J}} = [\tilde{F}]_{\mathcal{J}}$, also it results $q \simeq \{\tilde{F}_\vartheta\}_{\tilde{F} \in \tilde{\mathcal{Z}}}$

Remark.
The crisp set theory is the metatheory of the fuzzy set (object) theory: the quotient set $\tilde{\mathcal{Z}}/_{\mathcal{J}}$ is embedded into the metatheoretical environment.

3. Fuzzylike rough operators.

Let $\mathcal{E}$ be (indiscernibility) equivalence relation in Ω and π the $\mathcal{E}$-associated partition (with ω its elements).

Definition.
Let be: $r : \ddot{\mathcal{P}} \to \Lambda^{\pi}$ (by writing $r(\ddot{P}) = r_{\ddot{P}}$) s.t.

$$r_{\ddot{P}}(\omega) \begin{cases} = 1 & & \omega \in \underline{P} & \text{i/nternal condition} \\ \in]0,1[& \text{iff} & \omega \in \check{P} & \text{b/oundary (or indisc.)cond.} \\ = 0 & & \omega \notin \overline{P} & \text{e/xternal condition} \end{cases}$$

For any $x \in \Omega$, the value $r_{\ddot{P}}([x])$ is called the **rough membership grade** of x to respect $\ddot{P}$.

By e it's: $\omega \in \overline{P} \Leftrightarrow r_{\ddot{P}}(\omega) \neq 0$; hence we can to say:
$\ddot{P} = (\underline{P}, \overline{P}) = (\{\omega \in \pi \mid r_{\ddot{P}}(\omega) = 1\}, \{\omega \in \pi \mid r_{\ddot{P}}(\omega) \neq 0\})$.

The simbols $\neg, \sqcup, \sqcap$ let be, resp., the rough complementation operator, r. union op. and r.intersection op.; the simbols $\ast$ and $@$ are, resp., a t-norm and its dual t-conorm.

Definition.

By $\ddot{A}, \ddot{B} \in \ddot{\mathcal{P}}$, let be:

$\neg \ddot{A} = \ddot{C} = (\underline{C}, \overline{C})$ where $\forall \omega \in \pi$ it's $r_{\ddot{C}}(\omega) = 1 - r_{\ddot{A}}(\omega)$,

$\ddot{A} \sqcup \ddot{B} = \ddot{D} = (\underline{D}, \overline{D})$ " $r_{\ddot{D}}(\omega) = r_{\ddot{A}}(\omega) @ r_{\ddot{B}}(\omega)$,

$A \sqcap B = \ddot{E} = (\underline{E}, \overline{E})$ " $r_{\ddot{E}}(\omega) = r_{\ddot{A}}(\omega) \ast r_{\ddot{B}}(\omega)$.

Straightforwardly it follows the

Proposition.

$$\neg \ddot{A} = (\{\omega \in \pi \mid 1 - r_{\ddot{A}}(\omega) = 1\}, \{\omega \in \pi \mid 1 - r_{\ddot{A}}(\omega) \neq 0\}),$$
$$\ddot{A} \sqcup \ddot{B} = (\{\omega \in \pi \mid r_{\ddot{A}}(\omega) @ r_{\ddot{B}}(\omega) = 1\}, \{\omega \in \pi \mid r_{\ddot{A}}(\omega) @ r_{\ddot{B}}(\omega) \neq 0\}),$$
$$\ddot{A} \sqcap \ddot{B} = (\{\omega \in \pi \mid r_{\ddot{A}}(\omega) \ast r_{\ddot{B}}(\omega) = 1\}, \{\omega \in \pi \mid r_{\ddot{A}}(\omega) \ast r_{\ddot{B}}(\omega) \neq 0\}).$$

By algebricaly computing, it results:

Proposition.

1. $\underline{C} = \Omega - \overline{A}$, $\overline{C} = \Omega - \underline{A}$, $\underline{C} \subseteq \overline{C}$, $\check{C} = \check{A}$
2. $\underline{D} \supseteq \underline{A} \cup \underline{B}$, $\overline{D} = \overline{A} \cup \overline{B}$, $\underline{D} \subseteq \overline{D}$
3. $\underline{E} = \underline{A} \cap \underline{B}$, $\overline{E} \subseteq \overline{A} \cap \overline{B}$ $\underline{E} \subseteq \overline{E}$
4. $\neg(\ddot{A} \sqcup \ddot{B}) = \neg \ddot{A} \sqcap \neg \ddot{B}$
5. $\neg(\ddot{A} \sqcap \ddot{B}) = \neg \ddot{A} \sqcup \neg \ddot{B}$

Proof.

1. $\underline{C} = \{\omega \in \pi \mid 1 - r_{\ddot{A}}(\omega) = 1\} = \{\omega \in \pi \mid r_{\ddot{A}}(\omega) = 0\} = \{\omega \in \pi \mid \omega \notin \overline{A}\} = \Omega - \overline{A}$

$\overline{C} = \{\omega \in \pi \mid 1 - r_{\ddot{A}}(\omega) \neq 0\} = \{\omega \in \pi \mid r_{\ddot{A}}(\omega) \neq 1\} = \{\omega \in \pi \mid \omega \notin \underline{A}\} = \Omega - \underline{A}$

$\underline{A} \subseteq \overline{A} \Rightarrow \Omega - \underline{A} \supseteq \Omega - \overline{A}$

$\overline{C} - \underline{C} = (\Omega - \underline{A}) - (\Omega - \overline{A}) = \overline{A} - \underline{A}$

2. $\omega \in \underline{A} \cup \underline{B} \Rightarrow \omega \in \underline{A}$ or $\omega \in \underline{B} \Rightarrow r_{\ddot{A}}(\omega) = 1$ or $r_{\ddot{B}}(\omega) = 1 \Rightarrow r_{\ddot{A}}(\omega) \circledast r_{\ddot{B}}(\omega) = 1 \Rightarrow \omega \in \underline{D}$

$\omega \notin \bar{A} \cup \bar{B} \Rightarrow \omega \in \Omega - (\bar{A} \cup \bar{B}) = (\Omega - \bar{A}) \cap (\Omega - \bar{B}) \Rightarrow r_{\ddot{A}}(\omega) = r_{\ddot{B}}(\omega) = 0 \Rightarrow \omega \notin \bar{D}$

$\omega \notin \bar{D} \Rightarrow 0 = r_{\ddot{A}}(\omega) \circledast r_{\ddot{B}}(\omega) \geq r_{\ddot{A}}(\omega) \vee r_{\ddot{B}}(\omega) \Rightarrow r_{\ddot{A}}(\omega) = r_{\ddot{B}}(\omega) = 0 \Rightarrow \omega \notin \bar{A}, \omega \notin \bar{B}$

$\omega \in \underline{D} \Rightarrow r_{\ddot{D}}(\omega) = 1 \Rightarrow r_{\ddot{D}}(\omega) \neq 0 \Rightarrow \omega \in \bar{D}$

3. $\omega \in \underline{A} \cap \underline{B} \Rightarrow r_{\ddot{A}}(\omega) = r_{\ddot{B}}(\omega) = 1 \Rightarrow \omega \in \underline{E}$

$\omega \in \underline{E} \Rightarrow 1 = r_{\ddot{A}}(\omega) * r_{\ddot{B}}(\omega) \leq r_{\ddot{A}}(\omega) \wedge r_{\ddot{B}}(\omega) \Rightarrow r_{\ddot{A}}(\omega) = 1 = r_{\ddot{B}}(\omega) \Rightarrow \omega \in \underline{A} \cap \underline{B}$

$\omega \in \bar{E} \Rightarrow 0 \neq r_{\ddot{A}}(\omega) * r_{\ddot{B}}(\omega) \leq r_{\ddot{A}}(\omega) \wedge r_{\ddot{B}}(\omega) \Rightarrow r_{\ddot{A}}(\omega) \neq 0 \neq r_{\ddot{B}}(\omega) \Rightarrow \omega \in \bar{A} \cap \bar{B}$

$\omega \in \underline{E} \Rightarrow r_{\ddot{E}}(\omega) = 1 \Rightarrow r_{\ddot{E}}(\omega) \neq 0 \Rightarrow \omega \in \bar{E}$

4. $\neg(\ddot{A} \sqcup \ddot{B}) = (\{\omega \in \pi \mid 1 - r_{\ddot{A} \sqcup \ddot{B}}(\omega) = 1\}, \{\omega \in \pi \mid 1 - r_{\ddot{A} \sqcup \ddot{B}}(\omega) \neq 0\}) =$

$= (\{\omega \in \pi \mid r_{\ddot{A} \sqcup \ddot{B}}(\omega) = 0\}, \{\omega \in \pi \mid r_{\ddot{A} \sqcup \ddot{B}}(\omega) \neq 1\}) =$

$= (\{\omega \in \pi \mid r_{\ddot{A}}(\omega) \circledast r_{\ddot{B}}(\omega) = 0\}, \{\omega \in \pi \mid r_{\ddot{A}}(\omega) \circledast r_{\ddot{B}}(\omega) \neq 1\}) =$

$= (\{\omega \in \pi \mid 1 - (1 - r_{\ddot{A}}(\omega)) * (1 - r_{\ddot{B}}(\omega)) = 0\}, \{\omega \in \pi \mid 1 - (1 - r_{\ddot{A}}) * (1 - r_{\ddot{B}}(\omega)) \neq 1\}) =$

$= (\{\omega \in \pi \mid (1 - r_{\ddot{A}}(\omega)) * (1 - r_{\ddot{B}}(\omega)) = 1\}, \{\omega \in \pi \mid (1 - r_{\ddot{A}}(\omega)) * (1 - r_{\ddot{B}}(\omega)) \neq 0\}) =$

$= (\{\omega \in \pi \mid r_{\neg\ddot{A}}(\omega) * r_{\neg\ddot{B}}(\omega) = 1\}, \{\omega \in \pi \mid r_{\neg\ddot{A}}(\omega) * r_{\neg\ddot{B}}(\omega) \neq 0\}) = \neg\ddot{A} \sqcap \neg\ddot{B}$

5. Dually to 4.

4. Conclusions.

Let $\tilde{\pi} = \tilde{Z}_{\Lambda}(\Pi)$ be the zadehean of π . If $\ddot{P} = (\underline{P},\overline{P}) \in \ddot{\mathcal{P}}$, we considere the fuzzy sets $\tilde{\overline{P}}, \tilde{\underline{P}} \in \tilde{\pi}$ which membership function are $\mu_{\tilde{\overline{P}}} = r_{\ddot{P}}$ and $\mu_{\tilde{\underline{P}}} = r_{\ddot{P}}|_{\underline{P}}$. Then $\forall\ \ddot{P} \in \ddot{\mathcal{P}}$ let be:

$\tilde{Z}_{\ddot{P}} = \{\tilde{z} \in \tilde{\pi} \mid K_{\tilde{z}} = K_{\tilde{\underline{P}}},\ S_{\tilde{z}} = S_{\tilde{\overline{P}}}\}$.

The family $\{\tilde{Z}_{\ddot{P}}\}_{\ddot{P} \in \ddot{\mathcal{P}}}$ is a crisp partition of $\tilde{\pi}$ and $\mathcal{J}$ (of sect.2) is the associated equivalence relation.

By $\tilde{Z}_{\ddot{P}} = [\tilde{z}]_{\mathcal{J}}$ and by ϑ as in sect. 2, it's:

$$\ddot{\mathcal{P}} \simeq \tilde{\pi}/\mathcal{J} \simeq \{\tilde{F}_{\theta}\}_{\tilde{F} \in \tilde{\pi}} \quad .$$

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[1] N.U.Animobono - Finite rough sets as probabilistic-like fuzzy sets - BUSEFAL 34 (1988), 71-80