(α,β)-FUZZY MAPPINGS

S.K. BHAKAT
SIKSHA-SATRA
VISVA-BHARATI UNIVERSITY
and
P. DAS
DEPARTMENT OF MATHEMATICS
VISVA-BHARATI UNIVERSITY
SANTINIKETAN, WEST BENGAL
INDIA

1. INTRODUCTION

Fuzzy functions have been defined in several ways and some of them are incorporated in the book FUZZY SETS AND SYSTEMS : THEORY AND APPLICATIONS by Dubois and Prade[2].

So far as it is known that the idea of *belongingness* of a fuzzy point to a fuzzy set and *quasi-coincidence* of a fuzzy point with a fuzzy set has not been explicitly used in the existing definitions of fuzzy functions.

Let X and Y be two non-empty sets and $f: X \longrightarrow Y$ be a mapping. Let A and B be two subsets of X and Y respectively. f is called a mapping from A to B if $x \in A$ implies $f(x) \in B$.

If λ, μ are two fuzzy subsets of X and Y respectively, then f is said to have a *fuzzy domain* λ and a *fuzzy range* μ if and only if

(P): for all $x \in X$, $\mu(f(x)) \geq \lambda(x)$.

The condition (P) is equivalent to the following condition

(P'): for all $x \in X$ and $t \in (0,1]$, $x_t \in \lambda$ implies $f(x))_t \in \mu$.

The replacement of two "$\in$"^{s} in the condition (P') by any two of {$\in$, q, $\in\!\vee q$ (*belongs to or is quasi-coincidence with*), $\in\!\wedge q$ (*belongs to and is quasi-coincidence with*)} generates the concept of new types of fuzzy mappings, called *(α,β)-fuzzy mappings* from λ to μ where α or β stands for any one of {$\in$, q, $\in\!\vee q$, $\in\!\wedge q$} and $\alpha \neq \in\!\wedge q$.

Fuzzy injectivity and *surjectivity* of an *(α,β)-fuzzy mapping* are defined. The rest of the paper has been devoted to find the necessary and sufficient conditions for f to be an *(α,β)-fuzzy mapping* from λ to μ and to study the relation between ordinary injectivity (surjectivity) of the ordinary function f and the *fuzzy injectivity* (*or surjectivity*) of an (α,β)-*fuzzy mapping* from λ to μ.

Unless otherwise mentioned X, Y will denote two non-empty sets; f a mapping of X into Y; λ, μ fuzzy subsets of X and Y respectively with $\lambda(x) > 0$ for all $x \in X$. α or β will denote any one of $\{\in, q, \in\vee q, \in\wedge q\}$. $x_t\ \bar{\alpha}\ \lambda$ will mean that $x_t\ \alpha\ \lambda$ does not hold.

DEFINITION 1.1 f is said to be an (α,β)-*fuzzy mapping* $(\alpha \neq \in\wedge q)$ from λ to μ if
for all $x \in X$ and $t \in (0,1]$, $x_t\ \alpha\ \lambda$ implies $(f(x))_t\ \beta\ \mu$.

REMARK 1.2 The case $\alpha \neq \in\wedge q$ is omitted since there exist subsets λ s.t $\{x_t;\ x_t \in\wedge q\ \lambda\}$ is empty. In fact if $\lambda(x) \leq .5$ for all $x \in X$, then λ is such a fuzzy subset.

THEOREM 1.3 f is (q,q)-fuzzy map from λ to μ if and only if
for all $x \in X$, $\mu(f(x)) \geq \lambda(x)$.

THEOREM 1.4 Let f be an (α,β)-fuzzy map from λ to μ, where $(\alpha,\beta) = (\in,q), (\in,\in\wedge q), (q,\in), (q,\in\wedge q), (\in\vee q,\in), (\in\vee q,q), (\in\vee q,\in\wedge q)$.
Then $\mu(f(x)) = 1$ for all $x \in X$.

REMARK 1.5 The converse of the Theorem 1.4 is also true.

THEOREM 1.6 f is an $(\alpha,\in\vee q)$-fuzzy map from λ to μ, where $\alpha = \in, q, \in\vee q$ if and only if (i) $\lambda(x) \geq .5$ implies $\mu(f(x)) \geq .5$
(ii) $\lambda(x) < .5$ implies $\mu(f(x)) \geq \lambda(x)$.

DEFINITION 1.7 Let f be an (α,β)-fuzzy map from λ to μ.
f is said to be an (α,β)-*fuzzy injective* map from λ to μ if
for all $x,y \in X$, $x_{\lambda(x)}\ \bar{\alpha}\ y_{\lambda(x)}$ implies $(f(x))_{\lambda(f(x))}\ \bar{\beta}\ (f(y))_{\mu(f(y))}$.

THEOREM 1.8 Let f be an (α,β)-fuzzy map from λ to μ where $\alpha = \in, \in\vee q$. If f be injective then f is an (α,β)- fuzzy injective.

THEOREM 1.9 Let f be injective and let f be a (q,β)-fuzzy injective map from λ to μ.
(i) If $\beta = q$, then f is a (q,β)-fuzzy injective map from λ to μ if and only if for all $x \in X$, $\lambda(x) \leq .5$ implies $\mu(f(x)) \leq .5$,
(ii) If $\beta = \in, \in\vee q, \in\wedge q$, then f is a (q,β)-fuzzy injective map from λ to μ if and only if for all $x \in X$, $\lambda(x) > .5$.

REMARK 1.10 If $\beta = \in$, $\in\wedge q$, then since $\mu(f(x)) = 1$ for all $x \in X$ the condition (1) reduces to $\lambda(x) > .5$ for all $x \in X$.

THEOREM 1.11 Let f be an (α,β)-fuzzy injective map from λ to μ where $(\alpha,\beta) \neq (q,q)$. Then f is injective.

REMARK 1.12 Theorem 1.11 is not true if $(\alpha,\beta) = (q,q)$.

REMARK 1.13 The (q,q)-fuzzy injectivity of f implies the injectivity of f if $\mu(f(x)) > .5$ for all $x \in X$.

DEFINITION 1.14 An (α,β)-fuzzy map f from λ to μ is said to be an *(α,β)-fuzzy surjective map* if for all $y \in Y$, $t \in (0,1]$ such that $y_t \; \beta \; \mu$, then there exists $x \in f^{-1}(y)$ such that $x_t \; \alpha \; \lambda$.

REMARK 1.15 It follows from the Definition 1.14 that (α,β)-fuzzy surjectivity of f implies the surjectivity of f. But the converse is not true.

We denote for all $y \in Y$, $S_y = \sup\{\lambda(x);\ x \in f^{-1}(y)\}$.

THEOREM 1.16 Let f be an (α,β)-fuzzy surjective map from λ to μ.

(i) If $(\alpha,\beta) = (\in \in)$, (q,q), then

for all $y \in Y$, $\mu(y) \le S_y$,

(ii) If $(\alpha,\beta) = (\in\vee q, \in)$, $(\in\vee q, q)$, $(\in\vee q, \in\wedge q)$, then

for all $y \in Y$, $S_y \ge .5$,

(iii) If $(\alpha,\beta) = (\in, q)$, $(\in, \in\vee q)$, $(\in, \in\wedge q)$, $(q, \in)$, $(q, \in\vee q)$, $(q, \in\wedge q)$, then

for all $y \in Y$, $S_y = 1$,

(iv) If $(\alpha,\beta) = (\in\vee q, \in\vee q)$, then

for all $y \in Y$, $S_y < .5$ implies $\mu(y) = S_y$.

THEOREM 1.17 Let f be a surjective map of X onto Y. Let f be an (α,β)-fuzzy map from λ to μ where λ satisfies the "sup property". Then f is an (α,β)-fuzzy surjective map if

(i) for all $y \in Y$ and $(\alpha,\beta) = (\in, \in)$, (q,q), $\mu(y) \le S_y$

(ii) for all $y \in Y$ and $(\alpha,\beta) = (\in\vee q, \in)$, $(\in\vee q, q)$, $(\in\vee q, \in\wedge q)$, $S_y \ge .5$

(iii) for all $y \in Y$ and $(\alpha,\beta) = (\in, q)$, $(\in, \in\vee q)$, $(\in, \in\wedge q)$, $(q, \in)$, $(q, \in\vee q)$, $(q, \in\wedge q)$, $S_y = 1$

(iv) for all $y \in Y$ and $(\alpha,\beta) = (\in\vee q, \in\vee q)$, $S_y < .5$ implies $\mu(y) = S_y$.

DEFINITION 1.18 An (α,β)-fuzzy map from λ to μ is said to be (α,β)-*fuzzy bijective* if it is both (α,β)-fuzzy injective and (α,β)-fuzzy surjective.

THEOREM 1.19 Let $\lambda \in I^X$.
(i) f is an (α,α)-fuzzy map from λ to $f(\lambda)$.
(ii) f is injective. implies f is also an (α,α)-fuzzy injective map from λ to $f(\lambda)$.
(iii) If λ satisfies the "sup property", then f is an (α,α)-fuzzy surjective map from λ to $f(\lambda)$.

THEOREM 1.20 Let $\mu \in I^Y$. Then f is an (α,α)-fuzzy map from $f^{-1}(\mu)$ to μ. If f is injective (or surjective), then f is also an (α,α)-fuzzy injective (or surjective) map from λ to μ.

REMARK 1.21 The identity map of X is an (α,α)-fuzzy bijective map from λ to λ.

THEOREM 1.22 Let X, Y, Z be three sets and $\lambda \in I^X$, $\mu \in I^Y$, $\nu \in I^Z$. Let $f:X \longrightarrow Y$, $g:Y \longrightarrow Z$. If f is an (α,β)-fuzzy map from λ to μ and g, a (β,γ)-fuzzy map from μ to ν, then gof is an (α,γ)-fuzzy map from λ to ν. If f and g are (α,β) and (β,γ)-fuzzy injective (or surjective or bijective), then gof is also an (α,γ)-fuzzy injective (or surjective or bijective).

REMARK 1.23 Let f be bijective.
If f is an (α,β)-fuzzy map from λ to μ, where $\alpha = \beta$ and $\beta = \in\vee q$, then f^{-1} is not necessarily a (β,α)-fuzzy map from μ to λ.

THEOREM 1.24 Let f be bijective and f an (α,β)-fuzzy map from λ to μ. If f is (α,β)-fuzzy surjective from λ to μ, then f^{-1} is a (β,α)-fuzzy map from μ to λ.

THEOREM 1.25 Let X and Y be two groups and $f:X \longrightarrow Y$ a homomorphism. Let λ, μ be (α,β)-fuzzy subgroups of X and Y respectively.
If f is an (α,β)-fuzzy map from λ to μ, then $f^{-1}(\mu)$ is an (α,β)-fuzzy subgroup of X.

REFERENCES

1. Bhakat, S.K. & Das, P. : *On the definition of a fuzzy subgroup :* To be published in Fuzzy Sets And Systems.

2. Dubois, D. & Prade, H. : *Fuzzy Sets And Systems : Theory And Applications :* Academic Press (1980).

3. Ming, Pu Pao. & Ming, Liu Ying : *Fuzzy Topology I : Neighbourhood structure of a fuzzy point and Moore-Smith convergence :* J. Math. Anal. Appl. 76, (1980), 571-599.

4. Negoita, C.V. & Ralescu, D.A : *Application of Fuzzy sets to system analysis :* ISR. II. Birkhaeuser, Basel (1975) 18-24.

5. Rosenfled, A : *Fuzzy Groups :* J. Math. Anal. Appl. 35,(1971) 512-517.

6 . Zadeh, L.A : *Fuzzy Sets :* Inform and control 8,(1965) 338-353.

—— * ——