CHARACTERIZATIONS OF WEAKLY CONTINUOUS ORDER-HOMOMORPHISMS ON FUZZES

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ABSTRACT: The concept of weakly continuous order-homomorphisms on fuzzes was presented in [2]. In this paper, more characterizations with respect to weakly continuous order-homomorphisms are obtained by means of the θ-convergence of molecular nets, ideals on fuzzes.

KEYWORDS: order-homomorphism, fuzz, molecular net, ideal, weak continuity θ-closure

1. INTRODUCTION AND PRELIMINARIES

The notion of order-homomorphisms on fuzzes, which is a proper generalization of the concept of fuzz function [3] and is one of the most important tools of studying fuzzy topology and topology on fuzzes, i.e. so-called topological molecular lattices [4] as well, was first presented by G. J. Wang. Soon afterwards, he established and studied the concepts of continuous, closed and open order-homomorphisms on fuzzes [5]. In 1988, the author introduced and discussed the concepts of weaker forms of continuous order-homomorphisms, such as semi-continuity, almost continuity, weak continuity, S-continuity, O-continuity and so on, which are extensions of the corresponding concepts in [1]. The primary purpose of this paper is to give more characterizations of weakly continuous order-homomorphisms with the aid of the θ-convergence of molecular nets and ideals.

Throughout this paper L, L_1 will be fuzzes, i.e. completely distributive lattices with order-reversing involutions "$'$". M will be the set consisting of all moleculae [4], i.e. nonzero irreducible elements, of L. 0 and 1 will be denote the least and the greatest elements of L respectively. $(L(M), \delta)$ will denote a topological molecular lattice (briefly, TML) with the topology δ. For every $e \in M$, put $\eta(e) = \{P \in \delta' : e \nleq P\}$ and call the elements of $\eta(e)$ R-neighborhoods of e. Moreover, A^-, A° and A' will denote the closure, the interior and the pseudo-complement of $A \in L$ respectively.

DEFINITION 1.1 [5] A mapping $f: L \to L_1$ is called an order-homomorphism if the following conditions hold:

(H_1) $f(0)=0$.

(H_2) $f(\vee A_t)=\vee f(A_t)$ for $\{A_t\} \subset L$.

(H_3) $f^{-1}(B')=(f^{-1}(B))'$ for $B \in L_1$.

DEFINITION 1.2 [2] Let $f:(L(M),\delta)) \to (L_1(M_1),\delta_1)$ be an order-homomorphism; then

(1) f is called weakly continuous if for each $B \in \delta_1$, $f^{-1}(B) \leq (f^{-1}(B^-))^\circ$.

(2) f is called weakly continuous at $e \in M$ if for each $Q \in \eta(f(e))$, $(f^{-1}(Q^\circ))^- \in \eta(e)$.

THEOREM 1.1 [2] Let $f:(L(M),\delta) \to (L_1(M_1),\delta_1)$ be an order-homomorphism; then the following conditions are equivalent:

(1) f is weakly continuous.

(2) For each $Q \in \delta_1{}'$, $(f^{-1}(Q^\circ))^- \leq f^{-1}(Q)$.

(3) There exists a base β of δ_1 such that for each $F \in \beta'$, $(f^{-1}(F^\circ))^- \leq f^{-1}(F)$.

(4) There exists a base β of δ_1 such that for each $G \in \beta$, $f^{-1}(G) \leq (f^{-1}(G^-))^\circ$.

THEOREM 1.2 [2] An order-homomorphism $f:(L(M),\delta) \to (L_1(M_1),\delta_1)$ is weakly continuous if and only if for every $e \in M$, f is weakly continuous at e.

2. THE CHARACTERIZATIONS DEPICTED BY MOLECULAR NETS

DEFINITION 2.1 Suppose that $(L(M),\delta)$ is a TML, then $e \in M$ is in the θ-closure of $A \in L$ (write $e \leq A^{\theta-}$) if $A \nleq P^\circ$ for each $P \in \eta(e)$; A is called θ-closed if $A^{\theta-} \leq A$.

Evidently, if $e \leq A^-$, then $e \leq A^{\theta-}$, i.e. $A \leq A^{\theta-}$. Therefore A is θ-closed if and only if $A=A^{\theta-}$.

THEOREM 2.1 An order-homomorphism $f:(L(M),\delta)\to(L_1(M_1),\delta_1)$ is weakly continuous if and only if for each $A\in L$, $f(A^-)\leqslant(f(A))^{\theta-}$.

PROOF. *Necessity*: Assume that f is weakly continuous and $A\in L$. In order to verify $f(A^-)\leqslant(f(A))^{\theta-}$, we only need to prove that $f(e)\leqslant(f(A))^{\theta-}$ for each $e\in M$ with $e\leqslant A^-$. For this purpose, let $Q\in\eta(f(e))$, using Theorem *1.2* we have $(f^{-1}(Q^\circ))^-\in\eta(e)$. Hence $A\nleqslant(f^{-1}(Q^\circ))^-$ according to Theorem *2.2.5* in [*4*], and hence $f(A)\nleqslant Q^\circ$ for each $Q\in\eta(f(e))$. This shows that $f(e)\leqslant(f(A))^{\theta-}$ by Definition *2.1*.

Sufficiency: Let $e\in M$ and $Q\in\eta(f(e))$; then $f(e)\nleqslant Q$, equivalently, $e\nleqslant f^{-1}(Q)$. We asser that $(f^{-1}(Q^\circ))^-\in\eta(e)$. In fact, since $(f^{-1}(Q^\circ))^-\in L$,

$$f((f^{-1}(Q^\circ))^-)\leqslant(f(f^{-1}(Q^\circ)))^{\theta-}\leqslant(Q^\circ)^{\theta-}$$

in the light of the hypothesis. However, $(Q^\circ)^{\theta-}\leqslant Q$ (otherwise, there is a molecula $b\leqslant(Q^\circ)^{\theta-}$ with $b\nleqslant Q$. On account of $Q\in\delta_1{}'$, so $Q\in\eta(f(b))$, and so $Q^\circ\nleqslant Q^\circ$. This is impossible.) ,Therefore $(f^{-1}(Q^\circ))^-\leqslant f^{-1}(Q)$, thus $e\nleqslant(f^{-1}(Q^\circ))^-$, i.e. $(f^{-1}(Q^\circ))^-\in\eta(e)$. It follows from Theorem *1.2* that f is weakly continuous.

DEFINITION 2.2 Assume that N is a molecular net in $(L(M),\delta)$ and $e\in M$. If N is not eventually in P° for each $P\in\eta(e)$, then e is said to be a θ-limit point of N or call N θ-converge to e, in symbols $N^\theta\to e$. Put $\theta\text{-lim}N=\vee\{e\in M: N^\theta\to e\}$.

From this definition we readily obtain that $e\leqslant\theta\text{-lim}N$ if and only if $N^\theta\to e$.

THEOREM 2.2 Let $(L(M),\delta)$ be a TLM, $A\in L$ and $e\in M$; then $e\leqslant A^{\theta-}$ if and only if there is a molecular net N in A which θ-converges to e.

PROOF. Suppose that $e\leqslant A^{\theta-}$, then $A\nleqslant P^\circ$ for each $P\in\eta(e)$. Using Theorem *1.5.29* in [*6*], there is a molecula $N(P)$ in A satisfying $N(P)\nleqslant P^\circ$. Choose $N=\{N(P):P\in\eta(e)\}$, one easily sees that N is a molecular net in A which θ-converges to e. Conversely, grant that $N=\{N(n):n\in D\}$ is a molecular net in A which θ-converges to $e\in M$, then N is not eventually in P° for every $P\in\eta(e)$, i.e. there exists $n_0\in D$ such that $N(n)\nleqslant P^\circ$

whenever $n \geqslant n_0$ $(n \in D)$. Since $N(n) \leqslant A$ for each $n \in D$, so $A \nleqslant P^\circ$, and so $e \leqslant A^{\theta -}$ by Definition *2.1*.

THEOREM 2.3 An order-homomorphism $f: (L(M), \delta) \to (L_1(M_1), \delta_1)$ is weakly continuous at $e \in M$ if and only if for any molecular net N in L, $f(N) \xrightarrow{\theta} f(e)$ whenever $N \to e$.

PROOF. Necessity: Presume that f is weakly continuous at $e \in M$ and that $N = \{N(n): n \in D\}$ is a molecular net in L with $N \to e$. From Definition *1.2* $(f^{-1}(Q^\circ))^- \in \eta(e)$ for each $Q \in \eta(f(e))$, thus there is $n_0 \in D$ satisfying $N(n) \nleqslant (f^{-1}(Q^\circ))^-$ whenever $n \geqslant n_0$ $(n \in D)$. This imply $f(N(n)) \nleqslant Q^\circ$ as long as $n \geqslant n_0$. Consequently, $f(N)$ θ-converges to $f(e)$ by Definition *2.2*.

Sufficiency: If f is not weakly continuous at $e \in M$, then there is $Q \in \eta(f(e))$ such that $(f^{-1}(Q^\circ))^- \notin \eta(e)$, i.e. $e \leqslant (f^{-1}(Q^\circ))^-$. Hence according to Theorem *2.2.5* in [*4*], $f^{-1}(Q^\circ) \nleqslant P$ for every $P \in \eta(e)$, thereby there exists a molecula $N(P) \leqslant f^{-1}(Q^\circ)$ with $N(P) \nleqslant P^\circ$. Take $N = \{N(P): P \in \eta(e)\}$; it is clear that $N \to e$ and $f(N)$ does not θ-converge to $f(e)$. So the sufficiency holds.

THEOREM 2.4 For an order-homomorphism $f: (L(M), \delta) \to (L_1(M_1), \delta_1)$ from a *TML* $(L(M), \delta)$ to a *TML* $(L_1(M_1), \delta_1)$, the following conditions are equivalent:

(*1*) f is weakly continuous.

(*2*) For every $e \in M$ and every molecular net N in L, $f(N) \xrightarrow{\theta} f(e)$ whenever $N \to e$.

(*3*) For each molecular net N in L, $f(\lim N) \leqslant \theta\text{-}\lim f(N)$.

PROOF. (*1*) => (*2*): It follows from Theorem *1.2* and Theorem *2.3*.

(*2*) => (*3*): Assume that N is a molecular net. For the sake of checking $f(\lim N) \leqslant \theta\text{-}\lim f(N)$, we only need to verify that $f(e) \leqslant \theta\text{-}\lim f(N)$ for each $e \in M$ with $e \leqslant \lim N$, equivalently, $f(N) \xrightarrow{\theta} f(e)$ for each $e \in M$ satisfying $N \to e$. Therefore, Condition (*3*) follows from Condition (*2*).

(*3*) => (*1*): Let $Q \in \delta_1'$ and $e \leqslant (f^{-1}(Q^\circ))^-$; then there is a molecular net N in $f^{-1}(Q^\circ)$ such that $N \to e$ by Theorem *2.5.7* in [*4*]. Obviously, $f(N)$ is a molecular net in Q°. Because f is order-preserving, $f(e) \leqslant f(\lim N) \leqslant \theta\text{-}\lim f(N)$ according to Condition (*3*), i.e. $f(N)$

$^{\theta}\to f(e)$. Hence $f(e)<(Q^{\circ})^{\theta -}$ in the light of Theorem 2.2. But $(Q^{\circ})^{\theta -}<Q$ (See the proof in Theorem 2.3), so $e<f^{-1}(Q)$, and so $(f^{-1}(Q^{\circ}))^{-}<f^{-1}(Q)$. This shows that f is weakly continuous by Theorem 1.1.

3. THE CHARACTERS PORTRAIED BY IDEALS

DEFINITION 3.1 Let I be an ideal in $(L(M),\delta)$. $e\in M$ is said to be a θ-limit point of I if for each $P\in\eta(e)$, $P^{\circ}\in I$, in symbols $I^{\theta}\to e$. Write $\theta\text{-}\lim I=\vee\{e\in M: I^{\theta}\to e\}$.

DEFINITION 3.2 Let β be an ideal base in $(L(M),\delta)$; then $I(\beta)=\{A\in L:$ there exists $B\in\beta$ such that $B\geqslant A\}$ is an ideal in L, we call $I(\beta)$ the ideal generated by β and define $\theta\text{-}\lim\beta=\theta\text{-}\lim I(\beta)$.

The proofs of the following propositions are straightforward and are omitted.

PROPOSITION 3.1 [7] If I is an ideal in $(L(M),\delta)$ and $e\in M$, then $I^{\theta}\to e$ if and only if $e\leqslant\theta\text{-}\lim I$.

PROPOSITION 3.2 [7] Assume that $f:(L(M),\delta)\to(L_1(M_1),\delta_1)$ is an order-homomorphism and that I is an ideal in L; then the following statements hold:

(1) $f^{*}(I)=\{B\in L_1:$ there exists $A\in I$ such that for each $e\in M$ with $e\nleq A$, $f(e)\nleq B\}$ is an ideal in L_1.

(2) $(f(I'))'$ is an ideal base in L_1.

THEOREM 3.1 In any *TML* $(L(M),\delta)$, $e<A^{\theta -}$ if and only if there is an ideal I in L such that $A\notin I$ and $I^{\theta}\to e$, where $A\in L$ and $e\in M$.

PROOF. Let $e<A^{\theta -}$ and $P\in\eta(e)$; then $A\nleq P^{\circ}$. Take $\beta=\{P^{\circ}: P\in\eta(e)\}$, it is clear that β is an ideal base in L. Choose $I=I(\beta)=\{B\in L:$ there exists $P\in\eta(e)$ such that $P^{\circ}\geqslant B\}$; then $A\notin I$ and $I^{\theta}\to e$. Conversely, suppose that there is an ideal in L with $I^{\theta}\to e$ and $A\notin I$. Using Definition 3.1, $P^{\circ}\in I$ for each $P\in\eta(e)$. Since I is a lower set, $A\nleq P^{\circ}$, and so $e<A^{\theta -}$.

THEOREM 3.2 An order-homomorphism $f:(L(M),\delta)\to(L_1(M_1),$

δ_1) is weakly continuous at $e \in M$ if and only if for each ideal I in L with $I \to e$, $f^*(I)^\theta \to f(e)$.

PROOF. *Necessity*: Assume that f is weakly continuous at $e \in M$ and that I is an ideal in L satisfying $I \to e$. Then for each $Q \in \eta(f(e))$, $(f^{-1}(Q^\circ))^- \in \eta(e)$ by Definition 2.1 and so $f^{-1}(Q^\circ) \in I$ in the light of Definition 1.1 in [7]. Now we affirm that $Q^\circ \in f^*(I)$. In fact, since $e \nleq f^{-1}(Q^\circ)$ if and only if $f(e) \nleq Q^\circ$, $Q^\circ \in f^*(I)$ according to Proposition 3.2. Consequently, $f^*(I)^\theta \to f(e)$ using Definition 3.1.

Sufficiency: Let f be not weakly continuous at $e \in M$; then there is $Q \in \eta(f(e))$ with $(f^{-1}(Q^\circ))^- \notin \eta(e)$ or $e < (f^{-1}(e))^-$. Hence there is an ideal I in L so that $f^{-1}(Q^\circ) \notin I$ and $I \to e$ by Theorem 1.3 in [7]. Now we prove that $Q^\circ \notin f^*(I)$. First of all, we will verify that $f^*(I) \subset \{B \in L_1 : Q^\circ \nleq B\}$. Suppose that there is $B \in f^*(I)$ satisfying $Q^\circ \leq B$; then there exists $A \in I$ with $e \nleq A$ such that $f(e) \nleq B$ by the definition of $f^*(I)$, and so $f(e) \nleq Q^\circ$. This means that $e \leq A$ whenever $f(e) \leq Q^\circ$. Hence $f^{-1}(Q^\circ) \leq A$. Since I is a lower set and $A \in I, f^{-1}(Q^\circ) \in I$. It contradicts the definition of I. So $f^*(I) \subset \{B \in L : Q^\circ \nleq B\}$, and so $Q^\circ \notin f^*(I)$. This shows that $f(e)$ is not a θ-limit point of $f^*(I)$. Therefore the sufficiency is proved.

THEOREM 3.3 An order-homomorphism $f:(L(M),\delta) \to (L_1(M_1),\delta_1)$ is weakly continuous at $e \in M$ if and only if for each ideal I in L satisfying $I \to e$, $(f(I'))'^\theta \to f(e)$.

PROOF. *Necessity*: Let f be weakly continuous and I be an ideal in L with $I \to e$; then $(f^{-1}(Q^\circ))^- \in \eta(e)$ for each $Q \in \eta(f(e))$, and then $f^{-1}(Q^\circ) \in I$. Because $Q^{\circ\prime} \geq f(f^{-1}(Q^{\circ\prime}))$,

$$Q^\circ \leq (f(f^{-1}(Q^{\circ\prime})))' = (f((f^{-1}(Q^\circ))'))'$$

Put $A = f^{-1}(Q^\circ)$; then $A \in I$ and $Q^\circ \leq (f(A'))' \in (f(I'))'$. so $(f(I'))'^\theta \to f(e)$ in accordance with Definition 3.2.

Sufficiency: Grant that the condition is satisfied. If f is not weakly continuous at $e \in M$, then there exists $Q \in \eta(f(e))$ such that $e \leq (f^{-1}(Q^\circ))^-$, and then there is an ideal I in L with $I \to e$ and $f^{-1}(Q^\circ) \notin I$ by Theorem 1.3 in [7]. We asser that $f(e)$ is not a θ-limit point of $(f(I'))'$. In fact, on account of $f^{-1}(Q^\circ) \notin I$, so $f^{-1}(Q^\circ) \nleq A$, i.e. $Q^\circ \nleq (f(A'))'$ for each $A \in I$. This imply

$Q^\circ \notin (f(I'))'$. Therefore $f(e)$ is not a θ-limit point of $(f(I'))'$. However, it contradicts the hypothesis. Consequently, f must be weakly continuous at e.

THEOREM 3.4 Let $f:(L(M),\delta)\to(L_1(M_1),\delta_1)$ be an order-homomorphism; then the following conditions are equivalent:

(*1*) f is weakly continuous.

(*2*) For each $e\in M$ and each ideal I in L such that $I\to e$, $f^*(I)\overset{\theta}{\to}f(e)$.

(*3*) For each ideal I in L, $f(\lim I)\leqslant\theta\text{-}\lim f^*(I)$.

Proof. (*1*) => (*2*) : It follows from Theorem *1.2* and Theorem *3.2*.

(*2*) => (*3*) : Presume that I is an ideal in L, $e\in M$ and $f(e)\leqslant f(\lim I)$. With f being order-preserving, we have $e\leqslant\lim I$ or $I\to e$. Hence $f^*(I)\overset{\theta}{\to}f(e)$ follows from Condition (*2*). Thus $f(\lim I)\leqslant\theta\text{-}\lim f^*(I)$.

(*3*) => (*1*) : Let $Q\in\delta_1{}'$ and $e\leqslant(f^{-1}(Q^\circ))^-$; then there is an ideal I with $I\to e$ and $f^{-1}(Q^\circ)\notin I$ by Theorem *1.3* in [*7*]. Because of $f^*(I)\subset\{B\in L_1: Q^\circ\nleqslant B\}$, so $Q^\circ\notin f^*(I)$. According to Condition (*3*) we obtain $f(e)\leqslant f(\lim I)\leqslant\theta\text{-}\lim f^*(I)$. Therefore

$$f(e)\leqslant(Q^\circ)^{\theta-}\leqslant Q,\ \text{i.e.}\ e\leqslant f^{-1}(Q)$$

using Theorem *3.1*, and so $(f^{-1}(Q^\circ))^-\leqslant f^{-1}(Q)$. On account of Theorem *1.1* it follows that f is weakly continuous.

Analogous to the proof of Theorem *3.4*, we have the following result.

THEOREM 3.5 Let $f:(L(M),\delta)\to(L_1(M_1),\delta_1)$ be an order-homomorphism; then the following conditions are equivalent:

(*1*) f is weakly continuous.

(*2*) For each $e\in M$ and each ideal I with $I\to e$ in L, $(f(I'))'\overset{\theta}{\to}f(e)$.

(*3*) For any ideal I in L, $f(\lim I)\leqslant\theta\text{-}\lim(f(I'))'$.

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