A class of non–additive signed measures

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Abstract: In this paper we introduce the notion of signed λ–measures by extending the concept of g_λ–fuzzy measures to λ–measures and investigate its basic properties. Moreover, the decomposition theorems with respect to signed λ–measures are presented, such as the Hahn decomposition theorem, the Jordan decomposition theorem and the Lebesgue decomposition theorem, in the form of extension.

keywords: λ–measure; signed λ–measure; decompositon theorem; g_λ–fuzzy measure.

1. Introduction

Sugeno [6] introduced a class of non–additive set functions g_λ which was called g_λ–fuzzy measures or λ–additive fuzzy measures and applied it to treat complicated problems of engineering. Berres [1] showed that any λ–additive fuzzy measure can be represented by a density function as in probability theory. A relation between λ–additive fuzzy measures and measures was given by Kruse [3] and this relation was used to prove that any λ–additive fuzzy measure μ on a semiring **R** there exists an extension of μ to σ–algebra generated by **R**. And then Kruse [4] introduced a fuzzy integral with respect to λ–additive fuzzy measures which is the proper tool to express fuzzy expectations and gave a Radon–Nikodym–like theorem. In this paper we the first extend the set function's values from the unit interval [0,1] to the extended half–line $[0,+\infty]$, by introducing the concept of λ–measure. Then the definition of signed λ–measure is presented and some fundamental results concerning this notion, which are useful for the following discussion, are proved. Finally analogues of some of the important decomposition theorem in measure theory are established for signed λ–measures.

2. λ–measures and signed λ–measures

Throughout this paper, X denotes a nonempty set, **F** a σ–algebra on X and the pair (X,F) a measurable space. The extended half–line $[0,+\infty]$ and the extended real line $[-\infty,+\infty]$ are denoted simply by $\overline{R}_+$ and $\overline{R}$ respectively.

Definiton 1. Let (X, F) be a measurable space, let μ: $F \to \overline{R}_+$ be a set function, and let λ be a real number with $\lambda \in (-1/\sup\mu, +\infty)$ $(\sup\mu = \sup_{A\in F}\mu(A))$, $\lambda \neq 0$. μ is called a λ–measure on (X,F) (simply F) if it satisfies.

(1) $\mu(\phi)=0,$ (2.1)

(2) $\mu(\bigcup_{n=1}^{\infty} A_n)=\frac{1}{\lambda}[\prod_{n=1}^{\infty}(1+\lambda\mu(A_n))-1]$ (2.2)

for every sequence (A_n) of disjoint sets in F.

Since $-1/\sup\mu<\lambda$ and $\mu(A)/\sup\mu\leqslant 1$ (if $\mu\neq 0$), the inequalities

$$1+\lambda\mu(A)>1-\mu(A)/\sup\mu\geqslant 0$$

hold for each A in F. It follows that $1+\lambda\mu(\bigcup_{n=1}^{\infty} A_n)>0$ and so $\prod_{n=1}^{\infty}(1+\lambda\mu(A_n))\neq 0$, in (2.2) (precisely, if $\lambda<0$, $0<\prod_{n=1}^{\infty}(1+\lambda\mu(A_n))\leqslant 1$; if $\lambda>0$, $\prod_{n=1}^{\infty}(1+\lambda\mu(A_n))\geqslant 1$). Hence $[\prod_{n=1}^{\infty}(1+\lambda\mu(A_n))-1]/\lambda$ in (2.2) always exists, either as a nonnegative real number or as $+\infty$.

Evidently, (2.2) implies the following identity

$$\mu(\bigcup_{i=1}^{n} A_i)=\frac{1}{\lambda}[\prod_{i=1}^{n}(1+\lambda\mu(A_i))-1] \qquad (2.3)$$

for every finite sequence $A_1, \cdots\cdots, A_n$ of disjoint sets in F. In particular, for A, B$\in$F, $A\cap B=\phi$, we have

$$\mu(A\cup B)=\mu(A)+\mu(B)+\lambda\mu(A)\mu(B) \qquad (2.4)$$

and

$$\mu(A\cup B)=\mu(A)+(1+\lambda\mu(A))\mu(B) \qquad (2.5)$$

It follows immediately from (2.5) that a λ-measure on F is monotone, i. e., if $A\subset B$, $\mu(A)\leqslant\mu(B)$. In addition, in Theorem 1 and 2 we shall see that Sugeno's g_λ-fuzzy measures on F [6] are just λ-measures on F with $\mu(X)=1$, $\lambda\in(-1,+\infty)$, $\lambda\neq 0$.

We turn to the main definition of this paper.

Definition 2. Let (X, F) be a measurable space, let μ: $F\to\overline{R}$ be a set function, and let $\lambda(\neq 0)$ be a real number which satisfy.

(i) if $\inf\mu\geqslant 0$, $\lambda\in(-1/\sup\mu, +\infty)$;

(ii) if $\sup\mu\leqslant 0$, $\lambda\in(-\infty, -1/\inf\mu)$;

(iii) if $\sup\mu>0$ and $\inf\mu<0$, $\lambda\in(-1/\sup\mu, -1/\inf\mu)$

where $\inf\mu=\inf_{A\in F}\mu(A)$ and $\sup\mu=\sup_{A\in F}\mu(A)$. μ is called a signed λ-measure on F if it satisfies the conditions (2.1) and (2.2).

If $\mu(A)$ is finite for each $A\in F$, μ is said to be finite, and if there exists $(A_n)\subset F$ such that $\mu(A_n)$ is finite for each n and $X=\bigcup_{n=1}^{\infty} A_n$, μ is said to be σ-finite on F.

Let μ be a signed λ-measure on F. It are same proof as for λ-measures that $1+\lambda\mu(A)>0$ whenever $A\in F$ and $\prod_{n=1}^{\infty}(1+\lambda\mu(A_n))\neq 0$ for every infinite sequence (A_n) of disjoint set in F.

Suppose that μ is a signed λ–measure on the measurable space (X, F). Then the identities (2.3) (2.4) and (2.5) obviously hold. Moreover, for each A in F $\mu(A)+(1+\lambda\mu(A))\mu(A^c)$ must be defined (that is, must not be of the form $(+\infty)+(-\infty)$ or $(-\infty)+(+\infty)$) and must be equal $\mu(X)$. Hence if there is a set A in F for which $\mu(A)=+\infty$, then $\mu(X)=+\infty$, and if there is a set A in F for which $\mu(A)=-\infty$, then $\mu(X)=-\infty$. Consequently a signed λ–measure can not include both $+\infty$ and $-\infty$ among its values. A similar argument shows that if B is a set in F for which $\mu(B)$ is finite, then $\mu(A)$ is finite for each F–measurable subset A of B.

Example 1. If μ is a λ–measure on F, then $-\mu$ is a signed $-\lambda$–measure on F.

Theorem 1. A set function μ: $F\to\overline{R}$ is a signed λ–measure on F if and only if it satisfies (2.1) (2.4), and it is continuity from below. (that is, $\mu(\lim_{n\to\infty} A_n)=\lim_{n\to\infty}\mu(A_n)$ for every increasing sequence (A_n) of sets in F.)

Proof. If μ is a signed λ–measure on F, clearly μ satisfies (2.1) and (2.4). Suppose that (A_n) is an increasing sequence of set in F. Then by letting $A_0=\phi$

$$\mu(\lim_{n\to\infty} A_n)=\mu(\bigcup_{n=0}^{\infty}(A_{n+1}-A_n))=\frac{1}{\lambda}[\prod_{n=0}^{\infty}(1+\lambda\mu(A_{n+1}-A_n))-1]$$

$$=\lim_{n\to\infty}\frac{1}{\lambda}[\prod_{i=0}^{n}(1+\lambda\mu(A_{i+1}-A_i))-1]=\lim_{n\to\infty}\mu(A_n).$$

Hence, μ is continuity from below. Conversely, if μ satisfies (2.1) and (2.4), and it is continuity from below, then using the identity (2.4) we get the identity (2.3) by induction, and by using (2.3) and continuity from below we get (2.2), and this completes the proof of the theorem.

Example 2. If μ is a λ–measure on F, if υ is a $-\lambda$–measure on F, and if at least one of them is finite, then $\gamma=\mu-\upsilon-\lambda\mu\upsilon$ is a signed λ–measure on F.

In fact it is trivial to verify via Example 1 that γ satisfies all conditions in Theorem 1. We shall soon see that every signed λ–measure arises in this way.

The following theorem give two elementary but useful properties of signed λ–measures.

Theorem 2. Let μ be a signed λ–measure on F.

(1) if $A,B\in F$, $A\subset B$ and $|\mu(B)|<+\infty$, then $|\mu(A)|<+\infty$ and

$$\mu(B-A)=\frac{\mu(B)-\mu(A)}{1+\lambda\mu(A)} \tag{2.6}$$

(2) μ is continuity from above (that is, $\mu(\lim_{n\to\infty} A_n)=\lim_{n\to\infty}\mu(A_n)$ for every decreasing sequence (A_n) of sets in F, with $|\mu(A_n)|<+\infty$ for some n.)

Proof. First part of (1) have been proved in the comment behind Definition 2 and the equality (2.6) can be proved by (2.5).

Now suppose that (A_n) is a decreasing sequence of sets in F. and that $|\mu(A_n)|<+\infty$

holds for some n. We can assume that n = 1. Then we have the relations

$$A_1 = A_n \bigcup \left(\bigcup_{i=1}^{n-1} (A_i - A_{i+1})\right) \qquad n = 1,2\cdots$$

$$A_1 = \lim_{n\to\infty} A_n \bigcup \left(\bigcup_{i=1}^{\infty} (A_i - A_{i+1})\right)$$

It is easy to see that the sets A_1-A_2, A_2-A_3, $\cdots$, $A_{n-1}-A_n$, A_n are disjoint, and so are the sequence of set A_1-A_2, A_2-A_3, $\cdots$, $A_{n-1}-A_n$, $\lim_{n\to\infty} A_n$. Thus via (2.3) and (2.2) we get respectively

$$\mu(A_1) = \frac{1}{\lambda}[(1 + \lambda\mu(A_n))\prod_{i=1}^{n-1}(1 + \lambda\mu(A_i - \mu_{i+1})) - 1] \qquad n = 1,2\cdots$$

$$\mu(A_1) = \frac{1}{\lambda}[(1 + \lambda\mu(\lim_{n\to\infty} A_n))\prod_{i=1}^{\infty}(1 + \lambda\mu(A_i - A_{i+1})) - 1]$$

Let n approach infinity in the first resulting equality, we have

$$\mu(A_1) = \frac{1}{\lambda}[(1 + \lambda \lim_{n\to\infty} \mu(A_n))\prod_{i=1}^{\infty}(1 + \lambda\mu(A_i - A_{i+1})) - 1]$$

Using the last two equalities and the fact that $\prod_{i=1}^{\infty}(1 + \lambda\mu(A_i - A_{i+1})) \neq 0$, we find

$\mu(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$, and this completes the proof of (2).

There exists the following relationship between signed λ-measures and classical signed measures, which has given by Kruse for λ-additive fuzzy measures [3].

Theorem3. Let μ be a finite signed λ-measure on F. Define a real function θ_λ: [infμ, supμ]$\to \overline{R}$ by letting

$$\theta_\lambda(x) = \frac{\log(1 + \lambda x)}{\lambda}$$

Then the set function $\mu^* = \theta_\lambda \cdot \mu = \log(1+\lambda\mu)/\lambda$ is a finite signed measure on F. Converserly, if μ^* is a finite signed measure on F, then $\theta_\lambda^{-1} \cdot \mu^*$ is a finite signed λ-measure on F where $\theta_\lambda^{-1}(x) = (e^{\lambda x}-1)/\lambda$, $(\lambda \neq 0)$.

Proof. Suppose that μ is a finite signed λ-measure on F and (A_n) is a sequence of disjoint stes in F. Then the relation (2.2) implies that

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) = \frac{\log(1 + \lambda\mu(\bigcup_{n=1}^{\infty} A_n))}{\lambda} = \sum_{n=1}^{\infty} \frac{\log(1 + \lambda\mu(A_n))}{\lambda} = \sum_{n=1}^{\infty} \mu^*(A_n)$$

and this proved that μ^* is a signed measure on F since $\mu^*(\phi) = 0$. Obviously μ^* is finite.

Conversely, if μ^* is a finite signed measure on F. Define the set function μ: F$\to \overline{R}$ by letting

$$\mu = \theta_\lambda^{-1} \cdot \mu^* = \frac{e^{\lambda\mu^*} - 1}{\lambda} \qquad (\lambda \neq 0)$$

Then for every sequence (A_n) of disjoint sets in F

$$\mu(\bigcup_{n=1}^{\infty} A_n)=\frac{e^{\lambda\sum_{n=1}^{\infty}\mu^*(A_n)}-1}{\lambda}=\frac{1}{\lambda}[\prod_{n=1}^{\infty}(1+\lambda\mu(A_n))-1].$$

since $\mu(\phi)=0$. It follows that μ is a finite signed λ−measure on F.

3. Hahn and Jordan decomposition of signed λ−measures

In order to establish the Hahn decomposition of signed λ−measures, we first prove the following lemma.

Lemma 1. Let μ be a signed λ−measure on F. Then there is set C in F such that

$$\mu(C)=\inf_{A\in F}\mu(A)$$

Proof. Since the signed λ−measure μ cannot include both $+\infty$ and $-\infty$ among its values, we can for definiteness assume that $-\infty$ is not included. Let $\beta=\inf_{A\in F}\mu(A)$, and choose a sequence (A_n) of sets in F such that

$$\mu(A_n)<\beta\vee(-n)+\frac{1}{n}$$

hold for each n. Let $A=\bigcup_{n=1}^{\infty} A_n$, and constructe a sequence $\mathcal{A}_n$ of subclass by letting

$$\mathcal{A}_n=\{\bigcap_{i=1}^{n} A_i^*\ \text{: either}\ A_i^*=A_i\ \text{or}\ A_i^*=A-A_i\}$$

for each n. Then the sets of $\mathcal{A}_n$ form a finite partition of A in F, and the larger n is , the thiner the partition (that is, each set in $\mathcal{A}_n$ is the union of some sets in $\mathcal{A}_{n+1}$). Again constructe a sequence (B_n) of sets by letting

$$B_n=\bigcup_{\substack{c_i\in\mathcal{A}_n\\ \mu(c_i)<0}} C_i$$

Note that if E and F are disjoint sets in F and $\mu(F)<0$, then $\mu(E\bigcup F)\leqslant\mu(E)$ and $\mu(E\bigcup F)\geqslant\mu(F)$ via the equality (2.5). Hence for each n $\mu(A_n)\geqslant\mu(B_n)$. Moreover since $B_n\bigcup B_{n+1}$ is the union of B_n and some sets C_i in $\mathcal{A}_{n+1}$ which satisty $C_i\bigcap B_n=\phi$ and $\mu(C_i)<0$, then $\mu(B_n)\geqslant\mu(B_n\bigcup B_{n+1})$, likewise $\mu(B_n\bigcup B_{n+1})\geqslant\mu(B_n\bigcup B_{n+1}\bigcup B_{n+2})$. More generally the inequalities

$$\beta\vee(-n)+\frac{1}{n}>\mu(A_n)\geqslant\mu(B_n\bigcup B_{n+1}\bigcup\cdots\bigcup B_{n'}\geqslant\mu(\bigcup_{k=n}^{\infty} B_k)\geqslant\beta$$

Hold for each $n'>n$. Now define C by $C=\overline{\lim_{n\to\infty}} B^n$. Then C has the required property. In fact, continuity from above of μ and $|\mu(\bigcup_{k=n}^{\infty} B_k)|<+\infty$ impliy that $\mu(C)=\lim_{n\to\infty}\mu(\bigcup_{k=n}^{\infty} B_n)$

$= \beta$, and we simultaneously have proved that $\beta > -\infty$.

Theorem 4 (Extension of Hahn decomporition theorem).

Let μ be's signed λ–measure on (X, F). Then

(1) there are disjoint sets $A, B \in F$ such that $X = A \bigcup B$ and for an arbitrary F–measurable subset E of A(resp. B), $\mu(E) \geqslant 0$ (resp. $\mu(B) \leqslant 0$);

(2) if A_1 and B_1 also satisfy the condition above (1), then an arbitrary F–measurable subset of the symmetric difference $A_1 \Delta A$ (resp. $B_1 \Delta B$) has measure zero under μ.

Proof. (1) Let C have the required property in lemma 1 and let $B = C$, $A = \overline{C}$(the complement of C). Then for each F–measurable subset E of A,

$$\mu(E) = \frac{\mu(E \bigcup B) - \mu(B)}{1 + \lambda\mu(B)} \geqslant 0$$

since $\mu(B \bigcup E) \geqslant \beta$. Similarly, for each F–measurable subset E of B

$$\mu(E) = \frac{\mu(B) - \mu(B - E)}{1 + \lambda\mu(B - E)} \leqslant 0$$

since $\mu(B-E) \geqslant \beta$.

(2) If $E \subset A_1 - A$, on the one hand since $E \subset A_1$, $\mu(E) \geqslant 0$, on the other hand since $E \subset B$, $\mu(E) \leqslant 0$, therefore $\mu(E) = 0$. Likewise, if $E \subset A - A_1$, $\mu(E) = 0$.

It follows that for each F–measurable subset of $A_1 \Delta A$ has measure zero under μ. The same conclusion holds for $B_1 \Delta B$.

A pair (A,B) that satirfy the condition in (1) is called a Hahn decomposition of a signed λ–measure μ. Note that a signed λ–measrure can have several Hahn decompositions.

Example 3. If $X = \{a_1, a_2, a_3\}$, if F is the σ–algebra of all subsets of X, and if the signed λ–measure μ is defined by $\lambda = \frac{1}{2}$, $\mu(\{a_1\}) = 1$, $\mu(\{a_2\}) = 0$, $\mu(\{a_3\}) = -1$, and

$$\mu(A) = \frac{1}{\lambda}[\prod_{a_i \in A} (1 + \lambda\mu(a_i)) - 1]$$

for an arbitrary set A in F, then $(\{a_1, a_2\}, \{a_3\})$ and $(\{a_1\}, \{a_2, a_3\})$ are both Hahn decompositions of μ.

Corollary 1 (Extension of Jordan decomposition theorem)

If μ is a signed λ–measure on (X, F), if (A,B) is a Hahn decomposition of μ, if the set functions μ^+ and μ^- is defined by

$$\mu^+(E) = \mu(E \bigcap A) \qquad \text{and} \qquad \mu^-(E) = -\mu(E \bigcap B)$$

for each E in F, then

(1) μ^+ and μ^- do not depend on the particular Hahn decomposition used in their con struction.

(2) μ^+ is a λ–measure on F, $\mu^-$is a $-\lambda$–measure on F and at least one of them is finite, and

$$\mu = \mu^+ - \mu^- - \lambda\mu^+\mu^-$$

Proof.(1) Suppose that (A, B) and (A_1, B_1) are both Hahn decompositions of μ, then for each set E in F satisfies

$$E\cap A=[(A-A_1)\cap E]\cup(A\cap A_1\cap E)$$
$$E\cap A_1=[(A_1-A)\cap E]\cup(A\cap A_1\cap E)$$

Hence by Theorem 4 (2) and the relation (2.4), we get that $\mu(E\cap A)=\mu(E\cap A_1)$. Likewise $\mu(E\cap B)=\mu(E\cap B_1)$ holds for each E in F.

(2) It is clear that $\mu^+$is a λ-measure on F by Theorem 4(1), and by in addition a similar argument as in Example 1, μ^- is a $-\lambda$-measure on F. Since $+\infty$ and $-\infty$ cannot both occur among the values of μ at least one of μ^+ and μ^- must be finite. While the representation $\mu=\mu^+-\mu^--\lambda\mu^+\mu^-$ is a immediate consequence of the equality (2.4).

The representation $\mu=\mu^+-\mu^--\lambda\mu^+\mu^-$ is called the Jordan decomposition of the signed λ-measure μ. The variation $|\mu|$ of signed λ-measure μ, it is not sure to be a λ-measure, is defined by $|\mu|=\mu^++\mu^-+|\lambda|\mu^+\mu^-$. It is easy to show that the set function $|\mu|$ is monotone and continuous, and for $A\in F$, $|\mu|(A)=0$ if and only if $\mu^+(A)=\mu^-(A)=0$.

4. Lebesgue decomposition of signed λ-measures

Definition 3. Let μ and v be two signed λ-measures on (X, F). If $v(A)=0$ whenever $A\in F$ and $|\mu|(A)=0$, then v is said to be absolutely continuous with respect to μ, and denoted by $v\ll\mu$. If there is a set $N\in F$ such that $|\mu|(N)=0$ and for each $A\in F$ $|v|(A-N)=0$, then v is said to be singular with respect to μ, and denoted by $v\perp\mu$.

Theorem 5. Let μ and v be two signed λ-measures on (X, F). Then the following arguments are equivalent:

(i)$v\ll\mu$; (ii)$v^+\ll\mu$ and $v^-\ll\mu$; (iii)$|v|\ll|\mu|$.

The proof of Theorem 5 is similar to classical signed measure theory, and so is omitted.

In the sequel, we need the following lemma.

Lemma 2. Let μ be a λ-measure on F, and let (A_n) be a sequence of sets in F and satisfy $\lim\limits_{n\to\infty}\mu(A_n)=0$. Then there exists a subsequence (A_{ni}) of (A_n) such that $\mu(\overline{\lim\limits_{i\to\infty}} A_{ni})=0$.

Proof. Let ε be a positive number. Since $\lim\limits_{n\to\infty}\mu(A_n)=0$, we can construct a subsequence (A_{ni}) of (A_n) by choosing A_{n1} so that $\mu(A_{n1})<\frac{\varepsilon}{2}$and then choosing $A_{n2}(n_2>n_1)$ so that $(1+\lambda\mu(A_{n1}))\mu(A_{n2})<\frac{\varepsilon}{2^2}$(thus $\mu(A_{n1}\cup A_{n2})\leqslant\mu(A_{n1})+(1+\lambda\mu(A_{n1}))\mu(A_{n2})<\frac{\varepsilon}{2}+\frac{\varepsilon}{2^2}$) and in general, choosing the sets (A_{nj}) inductively so that $n_j>n_{j-1}$ and the relations

$$\mu(A_{n1}\cup A_{n2}\cup\cdots\cup A_{nj})<\frac{\varepsilon}{2}+\frac{\varepsilon}{2^2}+\cdots+\frac{\varepsilon}{2^j}$$

hold for j= 1,2,···, Thus we get a subsequence (A_{nj}) of (A_n) such that

$$\mu(\bigcup_{j=1}^{\infty} A_{nj}) = \lim_{m\to\infty} \mu(\bigcup_{j=1}^{m} A_{nj}) \leq \varepsilon$$

Hence, for $\varepsilon = 1$ there is a subsequence $(A_{ni}^{(1)})$ of (A_n) such that $\mu(\bigcup_{i=1}^{\infty} A_{ni}^{(1)}) < 1$. Note that $\lim_{i\to\infty} \mu(A_{ni}^{(1)}) = 0$ also holds for $(A_{ni}^{(1)})$, and so for $\varepsilon = \frac{1}{2}$ there is a subsequence $(A_{ni}^{(2)})$ of $(A_{ni}^{(1)})$ such that $\mu(\bigcup_{i=1}^{\infty} A_{ni}^{(2)}) < \frac{1}{2}$. In general, for each positive integer k there is a subsequence $(A_{ni}^{(k)})$ of $(A_{ni}^{(k-1)})$ such that $\mu(\bigcup_{i=1}^{\infty} A_{ni}^{(k)}) < 1/k$, where $(A_{ni}^{(0)})$ denotes (A_n).

Define a subsequence (A_{ni}) of (A_n) by $n_i = n_i^{(i)}$ for $i = 1,2,\cdots$. Obviously, (A_{ni}) satisfies

$$\bigcup_{i=k}^{\infty} A_{ni} \subset \bigcup_{i=1}^{\infty} A_{ni}^{(k)}$$

for $k = 1,2,\cdots$. and so the inequalities.

$$\mu(\overline{\lim_{i\to\infty}} A_{ni}) \leq \mu(\bigcup_{i=k}^{\infty} A_{ni}) \leq \mu(\bigcup_{i=1}^{\infty} A_{ni}^{(k)}) < \frac{1}{k}$$

hold for $k = 1,2,\cdots$, it follows that $\mu(\overline{\lim_{i\to\infty}} A_{ni}) = 0$, and the proof is complete.

Theorem 6. Let μ and v be two signed λ−measures on F and v be finite. Then $v \ll \mu$, if and only if for each positive ε there is a positive δ such that $|v|(A) < \varepsilon$, whenever $A \in F$ and $|\mu|(A) < \delta$.

Proof. the sufficiency is evident and we begin with the necessity.

Suppose that there is a positive number ε for which there is no suitable δ. For each integer n choose an $A_n \in F$ such that $|\mu|(A_n) < \frac{1}{n}$ hold for $n = 1,2,\cdots$. According to lemma 2 there is a subsequence (A_{ni}) of (A_n) such that $\mu^+ (\overline{\lim_{i\to\infty}} A_{ni}) = 0$ and $\mu^- (\overline{\lim_{i\to\infty}} A_{ni}) = 0$, movcover $|\mu|(\overline{\lim_{i\to\infty}} A_{ni}) = 0$. On the other hand since $|v|$ is monotone and finite

$$|v|(\overline{\lim_{i\to\infty}} A_{ni}) = \lim_{k\to\infty} |v|(\bigcup_{i=k}^{\infty} A_{ni}) \geq \overline{\lim_{k\to\infty}} |v|(A_{nk}) \geq \varepsilon$$

Thus the set $A = \overline{\lim_{i\to\infty}} A_{ni}$ satisfies $|\mu|(A) = 0$ but not $|v|(A) = 0$, and so $|v|$ is not absolutely continuous with respect to $|\mu|$, moreover so v not μ (See Theorem 5). This completes the proof of the theorem.

Theorem 7. Let μ and v be signed λ−measures on F. Then $v \perp \mu$, if and only if for each positive ε there is a set A in F such that $|\mu|(A) < \varepsilon$ and $|v|(A^c) < \varepsilon$.

Proof. The necessity is evident and we shall verify the sufficiency.

For each integer n choose a set A_n in F such that

$$|\mu|(A_n) < \frac{1}{n} \quad and \quad |v|(A_n^c) < \frac{1}{n}$$

the first inequality imply $\mu^+(A_n)) < \frac{1}{n}$ and $\mu^-(A_n) < \frac{1}{n}$, and so we get a sequence (A_n) of sets in F with $\lim_{n\to\infty} \mu^+(A_n) = 0$ and $\lim_{n\to\infty} \mu^-(A_n) = 0$. In view of lemma 2 there is a subsequence (A_{ni}) of (A_n) such that $\mu^+(\overline{\lim_{i\to\infty}} A_{ni}) = 0$ and $\mu^-(\overline{\lim_{i\to\infty}} A_{ni}) = 0$, furthermore, $|\mu|(\overline{\lim_{i\to\infty}} A_{ni}) = 0$. Since $|v|$ is monotone and continvity from below the relations

$$|v|((\overline{\lim_{i\to\infty}} A_{ni})^c) = |v|(\underline{\lim_{i\to\infty}} A_{ni}^c) = \lim_{k\to\infty} |v|(\bigcap_{i=k}^{\infty} A_{ni}^c) \leq \lim_{k\to\infty} |v|(A_{nk}^c) = 0$$

hold. Hence the set $N = \overline{\lim_{i\to\infty}} A_{ni}$ satisty $|\mu|(N) = 0$ and $|v|(A-N) = 0$ for each $A \in F$, i. e., $v \perp \mu$.

Theorem 8. Let μ and v be signed λ-measures on F. Then $v \ll \mu$ and $v \perp \mu$, if and only if $v(A) = 0$ whenever $A \in F$.

Proof. The sufficiency is clear and the necessity can be easily proved by the conditions $v \ll \mu$ and $v \perp \mu$ and the identity (2.4).

We can prove a similar and useful argument in which other conditions are used instead of v being a singed λ-measure in Theorem 8.

Lemma 3. Let v be a nonnegative set function on F with $v(A \bigcup B) = v(A)v(B)$, whenever $A, B \in F$ and $A \bigcap B = \phi$, and let μ be a signed λ-measure on F. If v and μ satisfy $(v-1) \ll \mu$, $(v-1) \perp \mu$, then $v(A) = 1$ hold for each A in F.

Proof. Using the condition $v-1 \perp \mu$ there is a set N in F such that $|\mu|(N) = 0$ and $v(A-N) = 1$ whenever $A \in F$. Since for each A in F $|\mu|(A \bigcap N) = 0$ and $v-1 \ll \mu$, $v(A \bigcap N) = 1$ hold for each $A \in F$. Therefore

$$v(A) = v(A-N) \bigcup (A \bigcap N) = v(A-N)v(A \bigcap N) = 1$$

hold for each A in F and the proof is completed.

Let us close this paper by introducing the Lebesgue decomposition theorem with respect to signed λ-measures.

Theorem 9 (Extension of Lebesgue decomposition theorem)
Let μ be a signed λ-measure on (X, F) and let v be a σ-finite signed λ-measure on (X, F). Then there are unique σ-finite signed λ-measures v_c and v_s on (X, F) such that the relations

$$v = v_c + v_s + \lambda v_c v_s, \quad v_c \ll \mu, \quad v_s \perp \mu$$

hold. The decomposition above is called the Lebesgue decomposition of v.

Proof. We begin with the case in which v is a finite signed λ-measure. Define R_μ by

$$R_\mu = \{A : A \in F \text{ and } |\mu|(A) = 0\}$$

It is easy to check that R_μ is a σ-subring of F. Since $|v|$ is monotone we can construct a sequence (N_n) of sets in R_μ by choosing N_1 in R_μ so that $|v|(N_1)=\sup\{|v|(A):A\in R_\mu, A\subset X\}$, and then choosing the remaining terms of (N_n) inductively so that the realtions $N_n\in R_\mu$ and $|v|(N_n)=\sup\{|v|(A):A\in R_\mu, A\subset X-\bigcup_{i=1}^{n-1}N_i\}$ hold for $n=2,3,\cdots$. Obviously, (N_n) is a sequence of disjoint sets in R_μ, and since $|v|$ is continuity from above the relations $\lim_{n\to\infty}|v|(N_n)\leqslant \lim_{n\to\infty}|v|(\bigcup_{i=n}^{\infty}N_i)=0$ hold and so $\lim_{n\to\infty}|v|(N_n)=0$. Let $N\setminus\bigcup_{n=1}^{\infty}N_n$ and define the set functions v_c, v_s by

$$v_c(A)=v(A-N),\quad v_s(A)=v(A\cap N),\quad A\in F.$$

It is clera that $|\mu|(N)=0$ and both v_c and v_s are finite signed λ-measures on F. Let us check that $v_s\perp\mu$ and $\lambda_c\ll\mu$. On the one hand for each A in F $|v|(A-N)\cap N)=0$, this implies $v_s\perp\mu$. On the other hand if $A\in F$ and $|\mu|(A)=0$, then $A\in R_\mu$, and according to the construction of N_n, the set A satisfies

$$|v|(A-N)=\lim_{n\to\infty}|v|(A-\bigcup_{i=1}^{n}N_i)\leqslant \lim_{n\to\infty}|v|(N_{n+1})=0$$

and $v_c(A)=0$ follows. This show $v\ll\mu$. In case v is a σ-finite signed λ-measure, let (X_n) be a partition of X into F-measureable sets that have finite measure under v, and for each n let $F_n=\{A\cap X_n : A\in F\}$. Then $F_n$is σ-algebra on X_n. Thus we can apply the construction above to the restrictions of the signed λ-measures μ and v to the spaces (X_n,F_n). Let N_1, N_2, $\cdots$ be the μ-mull subsets of X_1, X_2,$\cdots$ thus constructed and let $N=\bigcup_n N_n$.Then σ-finite signed λ-measures v_c and v_s defined by

$$v_c=v(A-N),\quad v_s(A)=v(A\cap N),\quad A\in F$$

form a Lebesgue decomposition of v.

We turn to the uniqueness of the Lebesgue decomposition. Let

$$v=v_c+v_s+\lambda v_c v_s,\quad v_c\ll\mu,\quad v_s\perp\mu;\qquad v=\bar{v}_c+\bar{v}_s+\lambda\bar{v}_c\bar{v}_s,\quad \bar{v}_c\ll\mu,\quad \bar{v}_s\perp\mu.$$

be two Lebesgue decompositions of v. First suppose that v is a finite signed λ-measure. Then the following identities hold

$$v=v_c+v_s+\lambda v_c v_s=\frac{1}{\lambda}[(1+\lambda v_c)(1+\lambda v_s)-1]$$

$$v=\bar{v}_c+\bar{v}_s+\lambda\bar{v}_c\bar{v}_s=\frac{1}{\lambda}[(1+\lambda\bar{v}_c)(1+\lambda\bar{v}_s)-1]$$

It follows that

$$\frac{1+\lambda v_c}{1+\lambda\bar{v}_c}=\frac{1+\lambda\bar{v}_s}{1+\lambda v_s}$$

Let $\gamma=\dfrac{1+\lambda v_c}{1+\lambda\bar{v}_c}$. It is easy to show that $\gamma(A\cap B)=\gamma(A)\gamma(B)$ whenever $A,B\in F$, $A\cap$

B= φ, and $(\gamma-1)\ll\mu$, $(\gamma-1)\perp\mu$. Therefore by lemma 3 $\gamma(A)=1$ hold for each A in F. This imply that $v_c=\bar{v}_c$ *and* $v_s=\bar{v}_s$. The case where v is a σ-finite singed λ-measure can be dealt with by similar was which has been used in the existence proof of the Lebesgue decomposition.

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