Some properties of fuzzy semi-topological elements

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Abstract: In this paper, first the concept of the semi-boundary of an element in a fuzzy to pological space is definition, and further investigate some properties of fuzzy semi- boundary element fuzzy semi-interior element and fuzzy semi- cloures element, and its application.

Throughout this paper L will denote a fuzzy lattice. Let 0, 1 denote the least element and the greatest element in L respactively. M will denote the collection of all molecules in L, therefore we also use L(M) to denote L. Let δ be a topology on L. Then (L, δ) is called a fuzzy topological space, or briefly fts. The elements of δ are called open elements and the elements of δ' are called closed elements, Where $\delta' = \{A' \mid A \in \delta\}$. Let

$$A^{o} = \vee \{B \in \delta \mid B \leqslant A\},$$

called the interior of A, and let

$$A^{-} = \wedge \{B \in \delta' \mid A \leqslant B\},$$

called the closure of A.

A element $B \in L$ is called a semi-open element of δ if there exists $O \in \delta$, Such that $O \leqslant B \leqslant O^{-}$, where O^{-} expresses the closure of O. FSO(L, δ) will denote the family of all semi-open elements in (L, δ). If O is open in (L, δ), then O is semi-open in (L, δ).

If B is semi-open element of δ then B' are called semi-closed element of δ. FSC(L, δ) will denote the family of all semi-closed element in (L, δ).

For each element A in L, let $A_{o} = \vee \{B \in FSO(L, \delta) \mid B \leqslant A\}$ called the semi-interior of A, and let $A\_ = \wedge \{B \in FSC(L, \delta) \mid A \leqslant B\}$ called the semi-closure of A.

Theorem 1 Let (L, δ) be a fts, $A, B \in L$, then

(1) $A \in FSO(L, \delta)$ iff $A = A_{o}$,

(2) $A \in FSC(L, \delta)$ iff $A = A\_$,

(3) $A\_\,\_ = A\_$, $A_{oo} = A_{o}$,

(4) if $A \leqslant B$, then $A_{o} \leqslant B_{o}$, $A\_ \leqslant B\_$,

(5) $A\_ \vee B\_ \leqslant (A \vee B)\_$,

(6) $(A\wedge B)_{-}\leqslant A_{-}\wedge B_{-}$,

(7) $A_{o}\vee B_{o}\leqslant(A\vee B)_{o}$.

(8) $(A\wedge B)_{o}\leqslant A_{o}\wedge B_{o}$.

Theorem 2 Let (L,δ) be a fts, $A\in L$, then

(1) $(A_{-})^{-}=(A^{-})_{-}=A^{-}$,

(2) $(A_{o})^{\circ}=(A^{\circ})_{o}=A^{\circ}$.

Theorem 3 Let (L,δ) be a fts, $A, B\in L$, then

(1) if $A_{-}=B_{-}$, then $A^{-}=B^{-}$,

(2) if $A_{o}=B_{o}$, then $A^{\circ}=B^{\circ}$.

The inverse of Theorem 3 is not valid in general.

Theorem 4 Let (L,δ) be a fts, $A, B\in L$, then

(1) if A or $B\in\delta'$, then $(A\vee B)_{-}=A_{-}\vee B_{-}$

(2) if A or $B\in\delta$, then $(A\wedge B)_{o}=A_{o}\wedge B_{o}$.

Definition 1 Let (L,δ) be a fts, $A\in L$, and put

$$[A]=\{a\in M \mid a\leqslant A_{-} \text{ and } a\nleqslant A_{o}\}$$

$$A_{\circ}=\vee[A].$$

Then the points in $[A]$ are called semi-boudary points of A, and $A_{\circ}$ is called the fuzzy semi-boundary of A.

Theorem 5 Let (L,δ) be a fts, $A\in L$, then

(1) $0_{\circ}=0$, $1_{\circ}=0$,

(2) $A_{\circ}\leqslant A_{-}$.

Proof. By Definition 1 it is clear.

Theorem 6 Let (L,δ) be a fts, $A\in L$. Then the following conditions are equivalent:

(1) $A_{\circ}=0$

(2) A is an semi-open and semi-closed element.

Proof . The conclusion is obvious.

Theorem 7 Let (L,δ) be a fts, $A\in L$. Then

(1) $A_{-}=A_{o}\vee A_{\circ}$,

(2) $A_{-}=A\vee A_{\circ}$.

Proof (1) This follows from Definition 1.

(2) By $A_{-}=A_{o}\vee A_{\circ}\leqslant A\vee A_{\circ}\leqslant A_{-}$.

Theorem 8 Let (L,δ) be a fts, $A\in L$. Then

(1) A is a semi-closed element iff $A_{\circ}\leqslant A$,

(2) A is a semi-open element iff $(A')_{\circ}\leqslant A'$.

Proof (1) suppose $A\in FSC(L,\delta)$, then $A=A_{-}=A\vee A_{\circ}$, by theorem 7 and hence $A_{\circ}\leqslant A$. conversely, suppose that $A_{\circ}\leqslant A$; then $A_{-}=A\vee A_{\circ}=A$ by theorem 7. Hence A is a closed element.

(2) The conclusion is obvious from (1).

Theorem 9 Let (L, δ) be a fts, $A \in L$. Then

(1) if $A_{\circ}$ is a semi-closed element, then $A_{\circ\circ} \leqslant A_{\circ}$,

(2) $A_e \leqslant A_{\circ}$,

(3) $A_{-\circ} \leqslant A_{\circ}$.

Proof. (1) By Theorem 8.

(2) If $A_e = 0$, then the conclusion is clear. If $A_e \neq 0$, then for each $a \in [A_o]$, We have $a \leqslant A_{o-} \leqslant A_-$ and $a \nleqslant A_{oo} = A_o$ Hence $a \leqslant A_{\circ}$ and $A_{o\circ} \leqslant A_{\circ}$

(3) If $A_{-\circ} = 0$, then the conclusion is clear. If $A_{-\circ} \neq 0$, then for each $a \in [A_-]$, We have $a \leqslant A_{--} = A_-$ and $a \nleqslant A_{-o}$ hence $a \nleqslant A_o$ by $a \nleqslant A_{--} = A_-$ and $A_o \leqslant A_{-o}$. Thus $a \leqslant A_{\circ}$ and $A_{-\circ} \leqslant A_{\circ}$.

Theorem 10 Let (L, δ) be a fts, A, B and $C \in L$. Then

(1) if $A \leqslant B$, then $A_{\circ} \leqslant B \vee B_{\circ}$,

(2) $C_- = C_{o-}$ iff $C_{\circ} \leqslant C_{o-}$.

Proof. (1) It follows that $A_{\circ} \leqslant A_- \leqslant B_- = B \vee B_{\circ}$ from Theorem 5, Theorem 7 and $A \leqslant B$.

(2) The necessity is clear by Theorem 7. Now suppose that $C_{\circ} \leqslant C_{o-}$, then $C_- = C_o \vee C_{\circ} \leqslant C_o \vee C_{o-} = C_{o-}$ by Theorem 7. Hence $C_- = C_{o-}$.

Definition 2 Let $f: (L_1, \delta_1) \rightarrow (L_2, \delta_2)$ be an order-homomoorphism (see[1]), If for each $B \in FSO(L_2, \delta_2)$, $f^{-1}(B) \in FSO(L_1, \delta_1)$, then f is said to be irresolute. If for each $A \in FSO(L_1, \delta_1)$, $f(A) \in FSO(L_2, \delta_2)$, then f is said to be semi-open. if for each $A \in FSC(L_1, \delta_1)$, $f(A) \in FSC(L_2, \delta_2)$, then f is said to be semi-closed.

Theorem 11 Let $f: (L_1, \delta_1) \rightarrow (L_2, \delta_2)$ be an order- homomorphism Then the following conditions are equivalent:

(1) f is irresolute,

(2) For each $C \in FSC(L_2, \delta_2)$, $f^{-1}(C) \in FSC(L_1, \delta_1)$,

(3) For each $A \in L_1$, $f(A_-) \leqslant (f(A))_-$,

(4) For each $B \in L_2$, $(f^{-1}(B))_- \leqslant f^{-1}(B_-)$,

(5) For each $B \in L_2$, $f^{-1}(B_o)_- \leqslant (f^{-1}(B))_o$,

Proof By definition 2 and Theorem 1 it is clear.

Theorem 12 Let $f: (L_1, \delta_1) \rightarrow (L_2, \delta_2)$ be an order- homomorphism Then the following condition are equivalent:

(1) f is irresolute,

(2) For each $A \in L_1$, $f(A_{\circ}) \leqslant (f(A))_-$.

(3) For each $B \in L_2$, $(f^{-1}(B))_{\circ} \leqslant f^{-1}(B_-)$.

Proof. (1) $\Longrightarrow$ (2) For each $A \in L_1$, it follows that $A_{\circ} \leqslant A_-$ by Theorem 5, hence $f(A_{\circ}) \leqslant f(A_-)$ by (1) and Theorem 11.

(2) ⟹ (3) By (2) and Theorem 1.1 in [3], We have $f((f^{-1}(B))_o) \leqslant (ff^{-1}(B))\_ \leqslant B\_$, and hence $(f^{-1}(B))_o \leqslant f^{-1}(B\_)$

(3) ⟹ (1) By virtue of theorem 7 and (3) it follows that $(f^{-1}(B))\_ = f^{-1}(B) \vee (f^{-1}(B))_o \leqslant f^{-1}(B) \vee f^{-1}(B\_) = f^{-1}(B\_)$, for each $B \in L_2$, and hence f is irresolute by theorem 11.

Theorem 13 Let $f: (L_1, \delta_1) \to (L_2, \delta_2)$ be an order- homomorphism. Then the following conditions are equivalent:

(1) f is semi-open,

(2) For each $A \in L_1$, $f(A_O) \leqslant (f(A))_O$.

(3) For each $A \in L_1$, $f(A_O) = (f(A_O))_O$

Proof (1) ⟹ (2). For each $A \in L_1$, it following that $f(A_O) \leqslant f(A)$ and hence $f(A_O) = (f(A_O))_O \leqslant (f(A))_O$ by (1).

(2) ⟹ (3) For each $A \in L_1$, we have $f(A_O) = f(A_{OO}) \leqslant (f(A_O))_O$ by (2), hence $f(A_O) = (f(A_O))_O$.

(3) ⟹ (1) For each $A \in FSO(L_1, \delta_1)$, it follows that $f(A) = f(A_O) = (f(A_O))_O \in FSO(L_2, \delta_2)$ by (3). Thus f is semi-open.

Theorem 14 Let $f: (L_1, \delta_1) \to (L_2, \delta_2)$ be an order -homomorphism Then the following conditions are equivalent.

(1) f is semi-closed.

(2) For each $A \in L_1$, $(f(A))\_ \leqslant f(A\_)$.

(3) For each $A \in L_1$, $f(A\_) = (f(A\_))\_$.

Proof (1) ⟹ (2) For each $A \in L_1$, it folloes that $(f(A))\_ \leqslant (f(A\_))\_ = f(A\_)$ on account of $A \leqslant A\_$ and (1).

(2) ⟹ (3) For each $A \in L_1$, we have $(f(A\_))\_ \leqslant f(A\_) = f(A\_)$ by (2) and hence $f(A\_) = (f(A\_))\_$.

(3) ⟹ (1) For each $A \in FSC(L_1, \delta_1)$, it follows that $f(A) = f(A\_) = (f(A\_))\_ \in FSC(L_2, \delta_2)$ by (3). Thus f is semi-closed.

References

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