SOME TOPOLOGICAL PROPERTIES OF L-MULTIFUNCTIONS

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In this paper new topological concepts connected with L-multifunctions theory are introduced and their properties are investigated. These concepts and properties are indispensable in connection with the analysis of L-economic systems.

Continuing our considerations on L-multifunctions we are using the abbrevations and notations from[1].

It is assumed that the reference spaces X,Y and Z are finite dimensional Euclidean spaces.

D e f i n i t i o n 1. An L-multifunction, $F:X\rightsquigarrow P(Y)$ say is called closed iff its graph W_F is a closed set.

C o r o l l a r y. For any closed L-multifunction its converse L-multifunction is closed.

D e f i n i t i o n 2. An L-multifunction, $a:X\rightsquigarrow P(Y)$ say, is sequentially bounded iff for any bounded sequence $S=\{x_n\}$ and any sequence $R=\{r_n\}$, $x_n\in X$, $r_n\in (0,1\rangle$, the set

$$\{(y,t)\in Y\times I : (x_n,y,r_n,t)\in W_F, x_n\in S, r_n\in R\}$$

is bounded.

T h e o r e m 1. If $F:X\rightsquigarrow P(Y)$ and $G:Y\rightsquigarrow P(Z)$ are closed L-multifunction and F is sequentially bounded L-multifunction, then $G\circ F$ is a closed L-multifunction.

P r o o f. Let $(x_n,z_n,r_n,p_n)\in W_{G\circ F}$ and let $(x_n,z_n,r_n,p_n)\to(x_0,z_0,r_0,p_0)$ as $n\to\infty$ (the convergence may be taken e.g. with respect to each coordinate separately), $x_n\in X$, $z_n\in Z$, r_n,p_n $(0,1\rangle$. We will prove that $(x_0,z_0,r_0,p_0)\in W_{G\circ F}$.
For any n (x_n,z_n,r_n,p_n) belongs to $W_{G\circ F}$ iff there exist $C_n\in Z$, $h_n\in I^Z$ such that $(x_n,C_n,r_n,h_n)\in G\circ F$ and $z_n\in C_n, h_n(z_n)=p_n$.
An element $(x_n,C_n,r_n,h_n)\in G\circ F$ iff there exist $(x_n,B_n,r_n,f_n)\in F$ and $(y_n,C_n,t_n,h_n)\in G$ such that $y_n\in B_n$ and $f_n(y_n)=t_n$. From the above conditions it follows that

$$(x_n,y_n,r_n,t_n)\in W_F \text{ and } (y_n,z_n,t_n,p_n)\in W_G.$$

Because F is a sequentially bounded and closed L-multifunction and G is a closed L-multifunction we observe that the sequences $\{y_n\}$ and $\{t_n\}$ are bounded and without losing generality we may assume that $y_n\to y_0$ and $t_n\to t_0$ as $n\to\infty$.
Moreover

$$(x_0,y_0,r_0,t_0)\in W_F \text{ and } (y_0,z_0,t_0,p_0)\in W_G.$$

This means that there exist $B_0\in P(Y)$, $f_0\in I^Y$ such that $y_0\in B_0$, $f_0(y_0)=t_0$, $(x_0,B_0,r_0,f_0)\in F$ and there exist $C_0\in P(Z), h_0\in I^Z$ such that $z_0\in C_0, h_0(z_0)=p_0$, $(y_0,C_0,t_0,h_0)\in G$. This means that $(x_0,C_0,r_0,h_0)\in G\circ F$.
Because $z_0\in C_0, h_0(z_0)=p_0$ so $(x_0,z_0,r_0,p_0)\in W_{G\ F}$.

T h e o r e m 2. If an L-multifunction $F:X\rightsquigarrow P(Y)$ is closed and for any $r,t\in I$ $(0,y,r,t)\notin W_F$ for $y\neq 0$, then F is a sequentially bounded L-multifunction.

P r o o f. According to Definition 2 it suffices to show that for any bounded sequence $S=\{x_n\}$ and any sequence $R=\{r_n\}$, $x_n\in X$, $r_n\in(0,1\rangle$ the set $T=\{(y,t)\in Y\times I: (x_n,y,r_n,t)\in W_F\}$ is bounded. Suppose that the set T is unbounded for some S and some R. Then there exist the sequences $\{y_n\}$, $\{t_n\}$, $(y_n,t_n)\in T$ such that $\|y_n\|\to\infty$ as $n\to\infty$. But $(x_n,y_n,r_n,t_n)\in W_F$ and F is a conical L-multifunction, so $(x_n/\|y_n\|, y_n/\|y_n\|, r_n, t_n)\in W_F$. Hence, there exist subsequences $x_{n_k}, y_{n_k}, r_{n_k}, t_{n_k}$ such that

$$(x_{n_k}/\|y_{n_k}\|, y_{n_k}/\|y_{n_k}\|\ \ r_{n_k}, t_{n_k})\to(0,y_0,r_0,t_0)$$

as $k\to\infty$, where $y_0\neq 0$ because $\lim_n \| y_{n_k}/\|y_{n_k}\| \| =1=y_0$.

Because F is a closed L-multifunction, so
$(0,y_0,r_0,t_0) \in W_F$ for $y_0 \neq 0$ - a contradiction .

D e f i n i t i o n 3. A fixed of the L-multifunction $F:X \to P(X)$ is an element $\bar{x} \in X$ such that there existe r,t from I such that $(\bar{x},\bar{x},r,t) \in W_F$.

T h e o r e m 3. (Fixed point theorem). Let C be a nonempty convex and compact subset of X. If $F:C \to P(C)$ is a closed, conical and superadditive L-multifunction, then F has a fixed point in C.

P r o o f . Let us consider a point -to-set mapping $\hat{F}:C \to P(C)$ such that for any $x \in C$

$$\hat{F}(x) = \{y \in C: \exists\, r,t \in I, \quad (x,y,r,t) \in W_F\}.$$

First we will prove that $\bar{x}$ is a fixed point of F iff $\bar{x}$ is a fixed point of $\hat{F}$. If $\bar{x}$ is a fixed point of $\hat{F}$, then $\bar{x} \in \hat{F}(\bar{x})$. This means there exist $r,t \in I$ such that $(\bar{x},\bar{x},r,t) \in W_F$, i.e. $\bar{x}$ is a fixed of F. Now, if $\bar{x}$ is a fixed point of F then from Definition 3 it follows that there exist r,t from I such that $(\bar{x},\bar{x},r,t) \in W_F$. This means that $\bar{x} \in \hat{F}(\bar{x})$, i.e. $\bar{x}$ is a fixed point for $\hat{F}$.
Now, we will show that $\hat{F}$ satisfies the hypothesis of Kakutani fixed point theorem, i.e. that $\hat{F}$ is a closed mapping and for any $x \in C$ $\hat{F}(x)$ is a convex set.

Convexity. Let y_1,y_2 be elements from $\hat{F}(x)$. From the definition of $\hat{F}$ it follows that there exist elements r_1,r_2,t_1,t_2 from I such that $(x,y_1,r_1,t_1) \in W_F$ and $(x,y_2,r_2,t_2) \in W_F$.
This means that there exist $B_1,B_2 \in P(C)$ and $f_1,f_2 \in I^C$ such that $(x,B_1,r_1,f_1) \in F$, (x,B_2,r_2,f_2) F and $y_1 \in B_1$, $y_2 \in B_2$, $f_1(y_1) = t_1$, $f_2(y_2) = t_2$. Because F is a conical and superadolitive L-multifunction, so for any $\alpha \geqslant 0$ $(x, \alpha B_1, +(1-\alpha)B_2, \min(r_1,r_2), \alpha f_1 + +(1-\alpha)f_2) \in F$.
This means, that
$(x, \alpha y_1 + (1-\alpha)y_2, \min(r_1,r_2), t) \in W_F$,
where $t = (\alpha f_1 + (1-\alpha)f_2)(\alpha y_1 + (1-\alpha)y_2)$,
i.e. $\alpha y_1 + (1-\alpha)y_2 \in \hat{F}(x)$.

Now, let us consider a sequence $\{x_n\}$, $x_n \in C$ such that $x_n \to x_0$ as $n \to \infty$. Let $y_n \in \hat{F}(x_n)$ and $y_n \to y_0$ as $n \to \infty$. We will prove that $y_0 \in \hat{F}(x_0)$. If $y_n \in \hat{F}(x_n)$ then for any n there exist $r_n, t_n \in I$ such that $(x_n, y_n, r_n, t_n) \in W_F$. Without losing generality we may assume that $r_n \to r_0$, $t_n \to t_0$ as $n \to \infty$. Because F is a closed L-multifunction, so $(x_0, y_0, r_0, t_0) \in W_F$. This means that $y_0 \in \hat{F}(x_0)$ i.e. $\hat{F}$ is a closed mapping. So, accordind to the Kakutani theorem there exists an element $\bar{x} \in C$ such that $\bar{x} \in \hat{F}(\bar{x})$. This means that an L-multifunction F has a fixed point $\bar{x}$ in C.

References

1 Matłoka, M.: Introduction and general properties of L-multifunctions, BUSEFAL (in point).