INTRODUCTION AND GENERAL PROPERTIES OF L - MULTIFUNCTIONS

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1. Introduction

The notion of Cartesian product plays an important role in the usual theory of functions and multifunctions. The Cartesian product of two fuzzy subset $A\in I^X$ and $B\in I^Y$ may be defined as the subset $A\times B$ of $X\times Y$ characterized by $(A\times B)(x,y) = \min(A(x),B(y))$. This definition has the inconvenience that when $A\times B$ is known and $A\times B \neq \emptyset$, it is impossible to retrieve again the subsets A and B. The notion of fuzzy Cartesian product (of ordinary sets and of fuzzy sets) which is introduced in paper[1]is free from this inconvenience. The L-multifunctions which we introduce in this paper are the subsets of a special case of Cartesian product and are also free from this inconvenience.

2. Definitions

Let X,Y and Z denote arbitrary but for further considerations fixed reference spaces. Next P(X), P(Y) and P(Z) denote respectively the families of all non - void subsets of X,Y and Z.

D e f i n i t i o n 1. An L-multifunction, $F:X\rightsquigarrow P(Y)$ say, is a subset of the Cartesian product $X\times P(Y)\times I\times I^Y$ satisfying the following conditions:

(i) If $(x,B,r,f)\in F$, then $r=0$ implies $B=\emptyset$ and $r=1$ implies $f(y)=1$ for any $y\in B$.

(ii) If $(x,B,r_1,f_1)\in F$ and $(x,B,r_2,f_2)\in F$, then $r_1>r_2$ implies $f_1\geqslant f_2$.

(iii) If $(x,B,r,f)\in F$ then supp $f=B$.

Let $\{x,r\}$ denote a fuzzy singleton in X with support x and value r. This fuzzy singleton is now transformed by F to a family of the fuzzy subsets in Y.

D e f i n i t i o n 2. A converse L - multifunction, F^{-1} say, to an L-multifunction $F:X\rightsquigarrow P(Y)$ is a subset of the Cartesian product $Y\times P(X)\times I\times I^X$ satisfying the following condition:

- $(y,A,t,h)\in F^{-1}$ if there exists $(x,B,r,f)\in F$ such that $x\in A$, $y\in B$, $f(y)=t$, $h(x)=r$.

D e f i n i t i o n 3. A composite, $G\circ F:X\rightsquigarrow P(Z)$ say, of two L-multifunctions $F:X\rightsquigarrow P(Y)$ and $G:Y\rightsquigarrow P(Z)$ is an L-multifunction such that

- $(x,C,r,h)\in G\circ F$ iff there exist $(x,B,r,f)\in F$ and $(y,C,t,h)\in G$ such that $y\in B, f(y)=t$.

Let X,Y and Z denote the linear spaces.

D e f i n i t i o n 4. An L-multifunction, $F:X\rightsquigarrow P(Y)$ say, is called conical iff for any $(x,B,r,f)\in F$ and for any $\alpha>0$ $(\alpha x,\alpha B,r,\alpha f)\in F$, where $(\alpha f)(y)=f(1/\alpha\, y)$ for any $y\in Y$.

D e f i n i t i o n 5. An L-multifunction, $F:X \rightsquigarrow P(Y)$ say is called superadolitive iff for any $(x_1,B_1,r_1,f_1)\in F$ and $(x_2,B_2,r_2,f_2)\in F$ $(x_1+x_2,\ B_1+B_2,\min\ (r_1,r_2),f_1+f_2)\in F$, where

$$(f_1+f_2)(y) = \sup_{y_1+y_2=y} \min\ (f_1(y_1),f_2(y_2)) \text{ for any } y\in Y.$$

3. Some properties of L-multifunctions.

T h e o r e m 1. If an L-multifunction is conical, then its converse L-multifunction is conical too.

P r o o f. As a matter of fact, let F be a conical L-multifunction. Let $(y,A,t,h)\in F^{-1}$. So, taking into account Definition 2.there exists $(x,B,r,f)\in F$ such that $x\in A, y\in B, f(y)=t$, $h(x)=r$. So, with respect to Definition 4 for any $\alpha>0$ we have $(\alpha x,\alpha B,r,\alpha f)$ F. Moreover $\alpha x\in\alpha A,\alpha y\in\alpha B,\alpha f(\alpha y)=f(y)=t,\alpha h(\alpha x)=h(x)=r$. This means that $(\alpha y,\alpha A,t,\alpha h)\in F^{-1}$. So, $F^{-1}$is a conical L-multifunction.

T h e o r e m 2. If F and G are conical L-multifunctions, then $G\circ F$ is a conical L- multifunction too.

P r o o f. Let $(x,C,r,h)\in G\circ F$. So, taking into account Definition 3 there exist $(x,B,r,f)\in F$ and $(y,C,t,h)\in G$ such that $y\in B, f(y)=t$. F and G are conical L-multifunctions,so for any $\alpha>0$ we have

$$(\alpha x,\alpha B,r,\alpha f)\in F \text{ and } (\alpha y,\alpha C,t,\alpha h)\in G.$$

Because $y\in B, f(y)=t$ so $\alpha y\in\alpha B$, $\alpha f(\alpha y) = f(y) = t$.

This means that

$(\alpha x, \alpha C, r, \alpha h) \in G \circ F$.

T h e o r e m 3. If an L-multifunction is superadolitive then its converse L-multifunction is superadolitive as well.

P r o o f. In point of fact, let an L-multifunction, $F: X \rightsquigarrow P(Y)$ say, satisfy the assumtion of the Theorem.
Let

$$(y_1, A_1, t_1, h_1)\ F^{-1} \text{ and } (y_2, A_2, t_2, h_2) \in F^{-1}$$

Then from the Definition 2. it follows that there exist

$$(x_1, B_1, r_1, f_1) \in F \quad \text{and } (x_2, B_2, r_2, f_2) \in F$$

such that

$$x_1 \in A_1,\ y_1 \in B_1,\ x_2 \in A_2,\ y_2 \in B_2,\ \ h_1(x_1) = r_1, h_2(x_2) = r_2,$$

$$f_1(y_1) = t_1\ ,\ \ f_2(y_2) = t_2.$$

We can assume that x_1, x_2, f_1 and f_2 are such that

$$h_1(x_1) = \sup_{x \in A_1} h_1(x) = r_1, \quad h_2(x_2) = \sup_{x \in A_2} h_2(x) = r_2$$

and

$$f_1(y_1) = \sup_{y \in B_1} f_1(y) = t_1\ ,\ f_2(y_2) = \sup_{y \in B_2} f_2(y) = t_2$$

Because F is superadolitive L-multifunction, so

$$(x_1 + x_2,\ B_1 + B_2,\ \min(r_1, r_2)\ ,\ f_1 + f_2\ ) \in F.$$

Moreover

$$x_1 + x_2 \in A_1 + A_2\quad ,\quad (h_1 + h_2)\ (x_1 + x_2) \overset{\min}{=} (r_1, r_2)$$

and

$$y_1 + y_2 \in\ B_1 + B_2\ ,\quad (f_1 + f_2)\ (y_1 + y_2) = \min(t_1, t_2).$$

This means that

$$(y_1+y_2,\ A_1+\dot{A}_2,\ \min(t_1,t_2),\ h_1+h_2) \in F^{-1}.$$

So, F^{-1} is superadditive L-multifunction.

T h e o r e m 4. If F and G are superadditive L-multifunctions then $G \circ F$ is a superdolitive L-multifunction too.

P r o o f. Let $(x_1,C_1,r_1,h_1) \in G \circ F$ and $(x_2,C_2,r_2,h_2) \in G \circ F$. Then , from the Definition 3 it follows that there exist

$$(x_1,B_1,r_1,f_1) \in F\ ,\ (y_1,C_1,t_1,h_1) \in G$$

such that $y_1 \in B_1$, $f_1(y_1) = t_1$

and there exist

$$(x_2,B_2,r_2,f_2) \in F\ ,\ (y_2,C_2,t_2,h_2) \in G$$

such that

$$y_2 \in B_2\ ,\ f_2(y_2) = t_2\ .$$

We can assume that

$$f_1(y_1) = \sup_{y \in B_1} f_1(y) = t_1$$

and

$$f_2(y_2) = \sup_{y \in B_2} f_2(y) = t_2$$

Because F and G are superadditive L-multifunctions, so

$$(x_1+x_2,\ B_1+B_2, \min\ (r_1,r_2)\ ,\ f_1+f_2) \in F$$

and

$$(y_1+y_2,\ C_1+C_2,\ \min\ (t_1, t_2),\ h_1+h_2) \in G$$

Moreover

$y_1+y_2\ B_1+B_2$ and $(f_1+f_2)(y_1+y_2) = \min(t_1,t_2)$.

This means that

$(x_1+x_2,\ C_1+C_2,\ \min(r_1,r_2),\ h_1+h_2) \in G \circ F$,

i.e. $G \circ F$ is superadditive L-multifunction.

D e f i n i t i o n 6. By the graph of an L-multifunction, $F: X \rightsquigarrow P(Y)$ say, it is understood a set W_F of the elements $(x,y,r,t) \in X \times Y \times I \times I$ such that there exist $B \in P(Y)$ and $f \in I^Y$ satisfying the following conditions:

- $y \in B$,
- $t = f(y)$,
- $(x,B,r,f,) \in F$.

D e f i n i t i o n 7. Let $\alpha,\beta \in I$. An α,β - cut of W_F, $W_F^{\alpha,\beta}$ in symbol, is a set of the elements $(x,y) \in X\ Y$ such that for $r \geqslant \alpha$ and $t \geqslant \beta$ $(x,y,r,t) \in W_F$.

T h e o r e m 5. If $F: X \rightsquigarrow P(Y)$ is a conical L-multifunction then for any α,β from I α,β - cut of W_F is a cone.

P r o o f. Let $(x,y) \in W_F^{\alpha,\beta}$. Then for $r \geqslant \alpha$ and $t \geqslant \beta$ $(x,y,r,t) \in W_F$. This means that there exist $B \in P(Y)$ and $f \in I^Y$ such that $y \in B$, $t=f(y)$ and $(x,B,r,f,) \in F$. Because F a conical L-multifunction, so for any $\lambda > 0$ $(\lambda x, \lambda B, r, \lambda f) \in F$. Moreover $\lambda y \in \lambda B$, $t = \lambda f(\lambda y) = f(y)$. This means that $(\lambda x, \lambda y, r, t) \in W_F$ and finally $(\lambda x, \lambda y) \in W_F^{\alpha,\beta}$.

T h e o r e m 6. If F is a conical and superadditive L-multifunction then for any α,β from I $W_F^{\alpha,\beta}$ is a convex set.

P r o o f. Let (x_1,y_1), $(x_2,y_2) \in W_F^{\alpha,\beta}$. Then for any $r_1,r_2 \geqslant \alpha$ and $t_1,t_2 \geqslant \beta$

$(x_1,y_1,r_1,t_1) \in W_F$, $(x_2,y_2,r_2,t_2) \in W_F$.

This means that there exist $B_1, B_2 \in P(Y)$ and $f_1, f_2 \in I^Y$ such that $y_1 \in B_1, t_1 = f_1(y_1)$,

$y_2 \in B_2$, $f_2(y_2) = t_2$ and $(x_1,B_1,r_1 f_1) \in F, (x_2,B_2,r_2,f_2) \in F$.

Because F is a conical and superadolitive L-multifunction, so for any $\lambda > 0$

$$(\lambda x_1, \lambda B_1, r_1, \lambda f_1) \in F \quad , ((1-\lambda)x_2, (1-\lambda) B_2, r_2, (1-\lambda) f_2 \in F$$

and

$$(\lambda x_1 + (1-\lambda)x_2, \lambda B_1 + (1-\lambda) B_2, \min(r_1,r_2), \lambda f_1 + (1-\lambda) f_2 \in F.$$

This means that

$$(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2, \min(r_1,r_2), \min(t_1,t_2)) \in W_F.$$

Because $\min(r_1,r_2) \geqslant \alpha$, $\min(t_1,t_2) \geqslant \beta$ so

$$(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \in W_F^{\alpha,\beta},$$

i.e. a set $W_F^{\alpha,\beta}$ is convex.

R e f e r e n c e s

[1] K.A.Dib, Nabil L. Youssel, Fuzzy Cartesian product, fuzzy relations and fuzzy functions, Fuzzy Sets and Systems 41 (1991) 299 - 315.