THE DISCUSSION ABOUT THE CONTAINMENT AND THE RANK OF REALIZABLE FUZZY MATRICES

Yi Chongxin Yuan Jie

Qiqihar Light-Chemical Engineering Institute, Qiqihar, CHINA

ABSTRACT

On the basis of references [1], [2], this paper further discusses the charactors of realizable fuzzy matrices and elementarily investigates relation between the rank and the containment of realizable fuzzy matrices.

Keywords : Realizable fuzzy matrix, Rank, Containment.

1 RELATIVE CONCEPTS

Assume that $\mathcal{M}_{n\times m}$ expresses a set consisting of all the n×m fuzzy matrices.

Definition 1.1 [1] Let $A\in\mathcal{M}_{n\times m}$, the space that is generated by the all row vectors of A is called row space of A and expressed as R(A). The number of vectors in the minimum generating basis of R(A) is called row rank of A and expressed as $P_r(A)$.

The column space c(A) and the column rank $P_c(A)$ of A may be similarly defined.

If $P_r(A)=P_c(A)=t$, then the number t is called rank of A and expressed as P(A).

Definition 1.2 [2] Let $A\in\mathcal{M}_{n\times m}$, the schein rank of A is the minimum number of the fuzzy matrices whose rank is all one and the sum of which is just equal to A and expressed as $P_s(A)$.

Definition 1.3 [1] Let $A=(a_1, \dots ,a_n)$, $B=(b_1, \dots ,b_m)$, $a_i,b_j\in[0,1]$ $(i=1, \dots ,n,\ j=1, \dots ,m)$, then $(A,B)=A^T\cdot B=(a_ib_j)_{n\times m}$ is called cross product multiplied A by B.

Definition 1.4 [2] Suppose that B is a $n\times n$ symmetrical fuzzy matrix, B is called the realizable fuzzy matrix if there exists a fuzzy matrix $A\in\mathcal{U}_{n\times m}$ making $B=A\cdot A^T$. The A hère is called a realizable matrix of B. And the number

$$r(B)=\min\left\{t \mid \exists A\in\mathcal{U}_{n\times t} \text{ and } A\cdot A^T=B\right\}$$

where $\mathcal{U}_{n\times t}$ is a set consisting of the all fuzzy matrices with n row vectors is called containment of B.

For the convenience of description in the following, we further suppose that $\mathcal{U}_R$ expresses the set consisting of all the $n\times n$ realizable fuzzy matrices in $\mathcal{U}_{n\times n}$ and do not distinguish sign "$\cdot$" from sign "$\wedge$" and sign "$+$" from sign "$\vee$", too.

2 THE RELATION BETWEEN THE CONTAINMENT AND THE RANK OF REALIZABLE FUZZY MATRICES

Theorem 2.1 Assume that $B\in\mathcal{U}_R$, then there exists rank P(B) of B.

Proof Suppose that B is separately divided into n pieces by row and column both like this :

$$B=(B_1, \dots, B_n)^T=(B_1^T, \dots, B_n^T).$$

If row vectors $B_{i_1}, \dots, B_{i_t}$ constitute a minimum generating basis of R(B), where $i_1, \dots, i_t$ 1, ..., n and which are not the same, then $P_r(B)=t$. Because B is a realizable fuzzy matrix, then $B=B^T$. Thus, $B_{i_1}^T, \dots, B_{i_t}^T$ form a minimum generating basis of C(B). Thereby $P_c(B)=t$. It follows that $P_r(B)=P_c(B)$ and there exists rank P(B) of B and $P(B)=t$.

Theorem 2.2 Assume that $B=(b_{ij})\in\mathcal{U}_R$, then $r(B)=1$ if any conclusion in the following is true :

1) $b_{ii}\wedge b_{jj}=b_{ij}$, for $\forall i,j\in\{1, \dots, n\}$

2) $A=(b_{11}, \ldots, b_{nn})^T$ is a realizable fuzzy matrix of B.

3) B may be expressed as below :

$$B=(b_1, \ldots, b_n)^T \cdot (b_1, \ldots, b_n)$$

4) $P_s(B)=P(B)=1$

Proof (circulating proof). Let $r(B)=1$, then exists a fuzzy vector $A=(a_1, \ldots, a_n)$ making

$$B=A \cdot A^T = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \cdot (a_1, \ldots, a_n) = \begin{bmatrix} a_1 a_1 & a_1 a_2 & \cdots & a_1 a_n \\ a_2 a_1 & a_2 a_2 & \cdots & a_2 a_n \\ \vdots & \vdots & & \vdots \\ a_n a_1 & a_n a_2 & \cdots & a_n a_n \end{bmatrix}$$

It is evident that elements in the main diagonal of B all take the form of $b_{ij}=a_i \wedge a_i$ $(1 \le i \le n)$. Thus, for all i,j $(1 \le i,j \le n)$, we have

$$b_{ij}=a_i \wedge a_j=(a_i \wedge a_i) \wedge (a_j \wedge a_j)=b_{ii} \wedge b_{jj}. \qquad (2.1)$$

And so the result 1) follows.

Suppose now that $b_{ij}=b_{ii} \wedge b_{jj}$, for all $i,j \in \{1, \ldots, n\}$, then

$$B=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} b_{11} b_{11} & b_{11} b_{22} & \cdots & b_{11} b_{nn} \\ b_{22} b_{11} & b_{22} b_{22} & \cdots & b_{22} b_{nn} \\ \vdots & \vdots & & \vdots \\ b_{nn} b_{11} & b_{nn} b_{22} & \cdots & b_{nn} b_{nn} \end{bmatrix}$$

$$=(b_{11}, \ldots, b_{nn})^T \cdot (b_{11}, \ldots, b_{nn}) . \qquad (2.2)$$

It expresses $A=(b_{11}, \ldots, b_{nn})^T$ is a realizable matrix of B.

If let $b_i=b_{ii}$ $(i=1, \ldots, n)$ then we have

$$B=(b_1, \ldots, b_n)^T \cdot (b_1, \ldots, b_n) . \qquad (2.3)$$

And so the result 3) follows. From the formula (2.2) it follows that

$$P_s(B)=P(B)=1 . \qquad (2.4)$$

On the contrary, from (2.4) we may back infer out the conclusions (2.3), (2.2) and (2.1) in sequence. Whereas from every one of preceding conclusions it follows that r(B)=1.

Theorem 2.3 Suppose that $B \in \mathcal{M}_R$ and r(B)=m, then B may be expressed the sum of m cross products.

Proof Let $A=(a_{ij})_{n\times m}$ be a realizable fuzzy matrix of B, then $B=A\cdot A^T$ and form formula

$$A\cdot A^T=\begin{pmatrix} \bigvee_{k=1}^{m} a_{1k}\ a_{1k} & \bigvee_{k=1}^{m} a_{1k}\ a_{2k} & \cdots & \bigvee_{k=1}^{m} a_{1k}\ a_{nk} \\ \bigvee_{k=1}^{m} a_{2k}\ a_{1k} & \bigvee_{k=1}^{m} a_{2k}\ a_{2k} & \cdots & \bigvee_{k=1}^{m} a_{2k}\ a_{nk} \\ \vdots & \vdots & & \vdots \\ \bigvee_{k=1}^{m} a_{nk}\ a_{1k} & \bigvee_{k=1}^{m} a_{nk}\ a_{2k} & \cdots & \bigvee_{k=1}^{m} a_{nk}\ a_{nk} \end{pmatrix}$$

$$=\begin{pmatrix} a_{11}a_{11} & a_{11}a_{21} & \cdots & a_{11}a_{n1} \\ a_{21}a_{11} & a_{21}a_{21} & \cdots & a_{21}a_{n1} \\ \vdots & \vdots & & \vdots \\ a_{n1}a_{11} & a_{n1}a_{21} & \cdots & a_{n1}a_{n1} \end{pmatrix} +\cdots+ \begin{pmatrix} a_{1m}a_{1m} & a_{1m}a_{2m} & \cdots & a_{1m}a_{nm} \\ a_{2m}a_{1m} & a_{2m}a_{2m} & \cdots & a_{2m}a_{nm} \\ \vdots & \vdots & & \vdots \\ a_{nm}a_{1m} & a_{nm}a_{2m} & \cdots & a_{nm}a_{nm} \end{pmatrix}$$

$$=\begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix}\cdot(a_{11}, \ \ldots\ , a_{n1}) + \ldots + \begin{bmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{bmatrix}\cdot(a_{1m}, \ \ldots\ , a_{nm})$$

It follows that B may be expressed the sum of m cross products.

Since $P_s(B)$ is the minimum number of cross products, the sum of which is just B. Thus we have :

Inference 2.1 Suppose that fuzzy matrix $B\in\mathcal{M}_R$, $P_s(B)\leq r(B)$ if the schein rank of B is $P_s(B)$.

Theorem 2.4 Assume that $B \in \mathcal{M}_R$ and the containment of B is r(B), then r(B) is a minimum number of realizable matrices whose containments are all one and the sum of which is just B.

Proof From the proof for the theorem 2.3, it follows that the B was expressed as the sum of r(B) realizable fuzzy matrices whose containment is all one. Thereinafter, we shall prove that the number t r(B) if the B expressed as the sum of t realizable fuzzy matrices whose containment is all one.

In fact, suppose that the B may be expressed as the sum of t realizable fuzzy matrices whose containment is all one. Thus we may let

$$B=\begin{bmatrix} c_{11} \\ \vdots \\ c_{n1} \end{bmatrix} \cdot (c_{11}, \ \dots \ ,c_{n1}) + \dots + \begin{bmatrix} c_{1t} \\ \vdots \\ c_{nt} \end{bmatrix} \cdot (c_{1t}, \ \dots \ ,c_{nt})$$

Hence, it is easy to work out.

$$B=\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1t} \\ c_{21} & c_{22} & \cdots & c_{2t} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nt} \end{bmatrix} \cdot \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & & \vdots \\ c_{1t} & c_{2t} & \cdots & c_{nt} \end{bmatrix} = C \cdot C^T$$

where $C=(c_{ij})_{n\ t}$. It follows that the matrix C is namely a realizable fuzzy matrix of B. In accordance with the definition of r(B), we may know that unequal t r(B) holds true.

REFERENCES

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2 Liu Wangtsin, The problem about realizability of fuzzy symmetrical matrix, Fuzzy mathematics, 1(1982), 69—76