LIMIT OF THE PRODUCT AND THE QUOTIENT OF COMPLEX FUZZY FUNCTIONS

YUE CHANGAN, LIU JIE AND LI JINHUI
HANDAN PREFECTURE EDUCATION COLLEGE
HANDAN, HEBEI, P.R.CHINA

In paper [1], we have given out the theorem of the limit operation of the sum and the difference of complex fuzzy functions. Now in this paper we would like to study the limit of the product and the quotient of complex fuzzy functions. First, we will give out the following lemmas.

Lemma 1 If $\lim_{x\to x_0} f_i(x) = A_i$, $A_i \in R$ (R is a set of the real number), $i=1,2,\cdots\cdots,n$, $\lim_{x\to x_0} g_j(x) = A$, $A \in R$, $j=1,2,\cdots\cdots m$, $A_i - A > 0$, both $fi(x)$ and $g_j(x)$ are the real value complex fuzzy functions, then there exists real number $\delta > 0$, so that when $d(x,x_0) < \delta$, we have

$$\inf\{f_i(x)\cdot g_j(x) \mid i=1,2,3,\cdots,n; j=1,2,3,\cdots m\}$$
$$= \inf\{g_j(x) \mid j=1,2,3,\cdots,m\}$$

Proof: Because

$$\lim_{x\to x_0} f_i(x) = A_i, \quad A_i \in R, \quad i=1,2,3,\cdots,n$$
$$\lim_{x\to x_0} g_j(x) = A, \quad A \in R, \quad j=1,2,3,\cdots,m$$
$$Ai - A > 0 \qquad i=1,2,3,\cdots,n$$

Therefore, we take

$$\varepsilon = \frac{1}{2}\inf\{A_i - A \mid i=1,2,3,\cdots,n\}$$

there exists $\delta > 0$, so that when $d(x,x_0) < \delta$, the following will be true:

$$d(f_i(x), A_i) = | f_i(x) - A_i |$$

$$< \frac{1}{2} \inf \{ A_i - A \mid i=1,2,3,\cdots,n \}$$

$$d(g_j(x), A) = | g_j(x) - A |$$

$$< \frac{1}{2} \inf \{ Ai - A \mid i=1,2,3,\cdots,n \}$$

so

$$A_i - \frac{1}{2} \inf \{ A_i - A \} < f_i(x) \qquad i=1,2,3,\cdots,n$$

$$g_j(x) < A + \frac{1}{2} \inf \{ A_i - A \} \qquad i=1,2,3,\cdots,n$$

$$j=1,2,3,\cdots,m$$

thus

$$A_i - \frac{1}{2} (A_i - A) < f_i(x) \qquad i=1,2,3,\cdots,n$$

$$g_j(x) < A + \frac{1}{2} (A_i - A) \qquad i=1,2,3,\cdots,n$$

$$j=1,2,3,\cdots,m$$

so that

$$f_i(x) > g_j(x) \qquad i=1,2,3,\cdots,n; \ j=1,2,3,\cdots,m$$

Finally, we take $\varepsilon = \frac{1}{2} \inf \{ A_i - A \mid i=1,2,3,\cdots,n \}$,there exists $\delta > 0$, so that when $d(x, x_0) < \delta$,we obtain:

$$\inf \{ f_i(x) g_j(x) \mid i=1,2,3,\cdots,n; j=1,2,3,\cdots,m \}$$

$$= \inf \{ g_j(x) \mid j=1,2,3,\cdots,m \}$$

Lemma2 If $\lim_{x \to x_0} g_j(x) = A$, $A \in R$ (R is a real number set), $j=1,2,3,\cdots m$, $g_j(x)$ is a real value complex fuzzy function, then

$$\lim_{x \to x_0} \inf \{ g_j(x) \mid j=1,2,3,\cdots,m \} = A.$$

Proof: From $\lim_{x \to x_0} g_j(x) = A, j=1,2,3,\cdots,m$, we know:

For $\varepsilon > 0$, there exists $\delta > 0$, so that when $d(x, x_0) < \delta$, the following will be true:

$$| g_j(x) - A | < \varepsilon . \qquad j=1,2,3,\cdots,m$$

Again, for x, there exists $k \in \{ 1,2,3\cdots,m \}$

so that

$$\inf\{g_j(x) \mid j=1,2,3,\cdots,m\} = g_k(x)$$

Thus, we have

$$\lim_{x\to x_0} \inf\{g_j(x) \mid j=1,2,3,\cdots,m\} = A.$$

Lemma 3 If $\lim_{x\to x_0} f_i(x) = A_i$, $A_i \in R, i=1,2,3,\cdots,n, \lim_{x\to x_0} g_j(x)=A$, $A \in R$, $j=1,2,3,\cdots,m$, R is a real number set, both $f_i(x)$ and $g_j(x)$ are real value complex fuzzy functions, $A-A_i>0$, $i=1,2,3,\cdots,n$, then there exists real number $\delta>0$, so that when $d(x,x_0)<\delta$, we have

$$\sup\{f_i(x)\cdot g_j(x) \mid i=1,2,3,\cdots,n; j=1,2,3,\cdots m\}$$

$$= \sup\{g_j(x) \mid j=1,2,3,\cdots,m\}$$

Proof: (leave out, for this proof is like the Proof of Lemma 1).

Lemma 4 If $\lim_{x\to x_0} g_j(x)=A$, $A \in R$ (R is a real number set), $j=1,2,3,\cdots,m$, $g_j(x)$ is a real value complex fuzzy function, then

$$\lim_{x\to x_0} \sup\{g_j(x) \mid j=1,2,3,\cdots,m\} = A$$

Proof: (leave out).

Lemma 5 If $\lim_{x\to x_0} g(x)=B$, $0 \bar{\in} [p[B],Q[B]]$ then $\lim_{x\to x_0}\frac{1}{g(x)}=\frac{1}{B}$.

Proof: According to the quotient definition of complex fuzzy number, for $0 \bar{\in} [a,b]$, we have

$$\frac{1}{[a,b]} = \left[\frac{1}{b},\frac{1}{a}\right], \quad \frac{1}{\underset{\sim}{[a,b]}} = \underset{\sim}{\left[\frac{1}{b},\frac{1}{a}\right]}$$

From theorem 3.5.3 in paper [3], if only we prove

$$\lim_{x\to x_0} P\left[\frac{1}{g(x)}\right] = P\left[\frac{1}{B}\right] = \frac{1}{Q[B]}$$

$$\lim_{x\to x_0} Q\left[\frac{1}{g(x)}\right] = Q\left[\frac{1}{B}\right] = \frac{1}{P[B]}$$

$$\lim_{x\to x_0} \inf\frac{1}{g(x)} = \inf\frac{1}{B} = \inf B$$

we can prove $\lim_{x\to x_0}\frac{1}{g(x)} = \frac{1}{B}$.

First, we prove: there exists $\delta > 0$, so that when $d(x,x_0) < \delta$,

$0 \bar{\in} [p[g(x)],Q[g(x)]]$ will hold.

Because $0 \bar{\in} [P[B],Q[B]]$, $\lim_{x\to x_0} g(x) = B$, therefore $\lim_{x\to x_0} P[g(x)] = P[B]$.

If $P[B]>0$, then, for $\varepsilon = P[B]$, there exists $\delta > 0$, so that when $d(x,x_0)<\delta$, we have

$$|P[g(x)]-P[B]| < P[B]$$

so $\quad 0<P[g(x)]<2P[B]$

That is, when $d(x,x_0)<\delta$, we have

$$0 \bar{\in} [P[g(x)],Q[g(x)]]$$

If $Q[B]<0$, take $\varepsilon = -Q[B]>0$, we will obtain the result in the same way.

Now, in $d(x,x_0)<\delta$, we will give out proofs of three public formulas of equal value.

First, we prove

$$\lim_{x\to x_0} P\left[\frac{1}{g(x)}\right] = P\left[\frac{1}{B}\right] = \frac{1}{Q[B]}$$

From the definition of the quotient of the complex fuzzy numbers, we know

$$\lim_{x \to x_0} P\left[\frac{1}{g(x)}\right] = \lim_{x \to x_0} \frac{1}{Q[g(x)]}$$

more over

$$\lim_{x \to x_0} \frac{1}{Q[g(x)]} = \frac{1}{\lim\limits_{x \to x_0} Q[g(x)]}$$

According to the equivalent of complex fuzzy space and the three—dimensional Euclid space, we have

$$\lim_{x \to x_0}[g(x)] = B$$

Therefore

$$\lim_{x \to x_0} P\left[\frac{1}{g(x)}\right] = P\left[\frac{1}{B}\right] = \frac{1}{Q[B]}$$

In the same way

$$\lim_{x \to x_0} Q\left[\frac{1}{g(x)}\right] = Q\left[\frac{1}{B}\right] = \frac{1}{P[B]}$$

Now, we can prove

$$\lim_{x \to x_0} \inf \frac{1}{g(x)} = \inf \frac{1}{B} = \inf B$$

Because $\lim g(x) = B$

Therefore, for $\varepsilon = 1 > 0$, there exists $\delta_1 > 0$, $\delta > \delta_1$, so that when $d(x, x_0) < \delta_1$, we have

$$d(g(x), B) = \max\{|P[g(x)] - P[B]|, |Q[g(x)] - Q[B]|, |\inf g(x) - \inf B|\} < 1$$

so

$$|\inf g(x)-\inf B|<1$$

Again, as $\inf g(x)$ and $\inf B$ take only one of the real numbers 0 or 1, therefore if $|\inf g(x)-\inf B|<1$, then

$$\inf g(x)=\inf B$$

thus
$$\lim_{x\to x_0}\inf g(x)=\inf B$$

That is, $\lim\limits_{x\to x_0}\inf\frac{1}{g(x)}=\inf\frac{1}{B}=\inf B$

Until now, proof end.

Theorem 1 If both f(x) and g(x) are the complex fuzzy functions and $\lim\limits_{x\to x_0} f(x)=A$, $\lim\limits_{x\to x_0} g(x)=B$, then $\lim\limits_{x\to x_0} f(x)g(x)=AB$.

Proof: The proof of this theorem can be changed into the proof of three equations, because $\lim\limits_{x\to x_0} f(x)g(x)=AB$ is equivalent to the following equality system:

$$\lim_{x\to x_0} P[f(x)g(x)]=P[AB]$$
$$\lim_{x\to x_0} Q[f(x)g(x)]=Q[AB]$$
$$\lim_{x\to x_0} \inf[f(x)g(x)]=\inf[AB]$$

First, it must be pointed out that when both f(x) and g(x) are real value fuzzy functions, we can prove this theorem by regarding them as three—element function of the classic integral calculus and to use the equivalent theory of the two kind of distance above limit concepts.

When f(x)(or g(x)) is not a real value function, first, we will prove the following formula:

$$\lim_{x\to x_0} P[f(x)g(x)]=P[AB].$$

Because

$$\lim_{x\to x_0} P[f(x)g(x)]$$
$$=\lim_{x\to x_0}\inf\{P[f(x)]P[g(x)],P[f(x)]Q[g(x)],Q[f(x)]P[g(x)],$$

$$Q[f(x)]Q[g(x)]\}$$

$$=\inf\{\lim_{x\to x_0} P[f(x)]P[g(x)],\ \lim_{x\to x_0} P[f(x)]Q[g(x)],\ \lim_{x\to x_0} Q[f(x)]P[g(x)],\ \lim_{x\to x_0} Q[f(x)]Q[g(x)]\}$$

$$=\inf\{\lim_{x\to x_0} P[f(x)]\lim_{x\to x_0} P[g(x)],\ \lim_{x\to x_0} P[f(x)]\lim_{x\to x_0} Q[g(x)],\ \lim_{x\to x_0} Q[f(x)]\lim_{x\to x_0} P[g(x)],\ \lim_{x\to x_0} Q[f(x)]\lim_{x\to x_0} Q[g(x)]\}$$

$$=\inf\{P[A]P[B],P[A]Q[B],Q[A]P[B],Q[A]Q[B]\}$$

$$=P[AB].$$

Therefore $\lim_{x\to x_0} P[f(x)g(x)]=P[AB]$.

In the same way

$$\lim_{x\to x_0} Q[f(x)g(x)]$$

$$=\lim_{x\to x_0} \sup\{P[f(x)]P[g(x)],P[f(x)]Q[g(x)],Q[f(x)]P[g(x)],Q[f(x)]Q[g(x)]\}$$

$$=\sup\{P[A]P[B],P[A]Q[B],Q[A]P[B],Q[A]Q[B]\}$$

$$=Q[AB]$$

Therefore $\lim_{x\to x_0} Q[f(x)g(x)]=Q[AB]$

Now, we will prove

$$\lim_{x\to x_0} \inf[f(x)g(x)]=\inf[AB].$$

Because $\lim_{x\to x_0} f(x)=A$, $\lim_{x\to x_0} g(x)=B$

Therefore $\lim_{x\to x_0} f(x)=\inf A$, $\lim_{x\to x_0} \inf g(x)=\inf B$.

Thus, for $\varepsilon=1>0$, there exists $\delta>0$, so that when $d(x,x_0)<\delta$, we have

$$|\inf f(x)-\inf A|<1,\quad |\inf g(x)-\inf B|<1$$

Again, as $\inf f(x)$, $\inf g(x)$, $\inf A$ and $\inf B$ take only one value of the real number 0 or 1, there must be $|\inf f(x)-\inf A|<1$ and $|\inf g(x)-\inf B|<1$, therefore

$$\inf A-1<\inf f(x)<\inf A+1$$

$$\inf B-1<\inf g(x)<\inf B+1$$

If $\inf A=0$, $\inf B=0$, then

$\inf f(x)=0$, $\inf g(x)=0$.

If $\inf A=1$, $\inf B=1$, then

$\inf f(x)=1$, $\inf g(x)=1$.

THus $\inf f(x)=\inf A$ $\inf g(x)=\inf B$

From the product definition of complex fuzzy numbers, we know

$$\inf[f(x)g(x)]=\inf[AB]$$

Therefore, for $\varepsilon>0$, there exists $\delta>0$, so that when $d(x,x_0)<\delta$, we have

$$|\inf[f(x)g(x)]-\inf[AB]|<\varepsilon$$

this is $\lim\limits_{x\to x_0}\inf[f(x)g(x)]=\inf[AB]$

Therefore, we obtain

$$\lim_{x\to x_0} f(x)g(x)=AB .$$

Theorem 2 If $\lim\limits_{x\to x_0} f(x)=A$, $\lim\limits_{x\to x_0} g(x)=B$, $0\overline{\in}[P[B],Q[B]]$

then $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)}=\frac{A}{B}$.

Proof: According to Lemma 5 and Theorem 1, we know

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0} f(x)\frac{1}{g(x)}=\lim_{x\to x_0} f(x)\lim_{x\to x_0}\frac{1}{g(x)}=A\cdot\frac{1}{B}=\frac{A}{B}.$$

Therefore $\lim\limits_{x\to x_0}\frac{f(x)}{g(x)}=\frac{A}{B}$.

Reference

[1] Yue Changan, Liu Jie, "Limit of the sum and the difference of complex fuzzy functions" ,n°50 issue 92 of BUSEFAL .

[2] Wang Peizhuang, "The theory of fuzzy set and its applications" ,Shang Hai Science Publishing House, 1983.

[3] Wu Heqin, Yue Changan, "The guide theory of the grey mathematics" ,HeBei People Publishing House,1990 .