Some Spaces of Sequences of F-numbers

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In this paper we introduce some spaces of sequences of fuzzy subsets on m-dimensional Euclidean space which are called F-numbers. We discuss spaces of bounded sequences, spaces of convergent sequences, l_1-spaces of sequences and spaces of all sequences, of F-numbers and show that they are all complete metric space.

Keywords. Sequences of F-numbers, bounded and convergent sequences of F-numbers, complete metric space.

Let R^m denote m-dimensional Euclidean space, A and B be two nonempty bounded subsets of R^m. The distance between A and B is defined by Hausdorff metric

$$d_H(A,B) = \text{Max}[\sup_{a\in A}\inf_{b\in B}\|a-b\| , \sup_{b\in B}\inf_{a\in A}\|a-b\|]$$

Where $\|\cdot\|$ denotes the usual euclidean norm in R^m.

<u>**Lemma 1.**</u> (see [1] theorem 2.1)
Let $Q(R^m)$ denote the set of all nonempty, compact subsets of R^m. Then $(Q(R^m), d_H)$ is a complete metric space.

<u>**Definition 1.**</u>
A fuzzy subset u: $R^m \to [0,1]$ with the following properties:
(a) $\{x\in R^m : u(x)\geq r\}$ is compact for each $r>0$,
(b) $\{x\in R^m : u(x)=1\}$ is nonempty.
is called a F-number.

We denote the set of all F-numbers by F^*.

<u>**Definition 2.**</u>
A sequence $X=\{X_n\}$ of F-numbers is a function X from the set N of all positive integers into F^*. The F-number X_n denotes the value of the function at $n\in N$.

Define a map d: $F^*\times F^* \to R$ by

$$d(u,v) = \sup_{r>0} d_H(L_r(u), L_r(v))$$

Where d_H is the Hausdorff matric and we denote by $L_r(w)=\{x: w(x)\geq r\}$ for $w\in F^*$.

<u>**Lemma 2.**</u> (see [1] theorem 4.1)
(F^*, d) is a complete metric space.

We now introduce some spaces of sequences of F-numbers. We have

$$b = \left\{X=\{X_n\} : \sup_n d(X_n, 0) < \infty\right\}$$

$$c = \left\{ X=\{X_n\} \ : \ \text{there exists } Y \in F^* \text{ s.t. } \ d(X_n, Y) \to 0 \right\}$$

$$c_0 = \left\{ X=\{X_n\} \ : \ d(X_n, 0) \to 0 \right\}$$

$$l_p = \left\{ X=\{X_n\} \ : \ \Sigma_n [d(X_n, 0)]^p < \infty \right\} \qquad (1 \leq p < \infty)$$

and denote the set of all sequences of F-numbers by s .

We have the following results.

Theorem 1.
b is a complete metric space with the metric f defined by

$$f(X, Y) = \operatorname{Sup}_n d(X_n, Y_n)$$

Where $X=\{X_n\}$ and $Y=\{Y_n\}$ are sequences of F-numbers which are in b.

Proof. It is straightforward to see that f is a metric on b. To show that b is complete in this metric, let $\{X^i\}$ be a Cauchy sequence in b. Then for each fixed n, $\{X_n^i\}$ is a Cauchy sequence in F^*. But F^* is complete with the metric d, there exists X_n in F^* such that $\operatorname{Lim}_i X_n^i = X_n$ for every n. Put $X=\{X_n\}$, we shall show that $\operatorname{Lim}_i X^i = X$ and $X \in b$. Since $\{X^i\}$ is a Cauchy sequence in b, given $\varepsilon>0$ there exists $k \in N$ such that for $i, j > k$ and every $n \in N$

$$d(X_n^i, X_n^j) < \varepsilon$$

Taking the limit as $j \to \infty$, we get $d(X_n^i, X_n) \leq \varepsilon$

Therefore $f(X^i, X) = \operatorname{Sup}_n d(X_n^i, X_n) \leq \varepsilon$ i.e. $\operatorname{Lim}_i X^i = X$.

From $f(X,0) \leq f(X, X^i) + f(X^i, 0)$ we obtain $\operatorname{Sup}_n d(X_n, 0) = f(X, 0) < \infty$ .So $X=\{X_n\} \in b$.

The proof is completed.

Theorem 2.
c is a complete metric space with the metric f defined by

$$f(X, Y) = \operatorname{Sup}_n d(X_n, Y_n)$$

Where $X=\{X_n\}$ and $Y=\{Y_n\}$ are sequences of F-numbers which are in c.

Proof. It is clear that (c,f) is a metric space. To prove the completeness of c, let $\{X^i\}$ be a Cauchy sequence in c. Repeating the proof of the theorem 1, we know that there exists $X=\{X_n\}$ such that $\operatorname{Lim}_i X^i = X$. We now show that $X \in c$.

Given $\varepsilon>0$, for each fixed $n \in N$ and enough large i

$$d(X_n^i, X_n) < \varepsilon$$

Since $\{X_n^i\} \in c$, for each fixed i, there exists $X_0^i \in F^*$ such that for enough large n

$$d(X_n^i, X_0^i) < \varepsilon$$

Since $\{X^i\}$ is a Cauchy sequence in c, for enough large i,j

$$d(X_n^i\ X_n^j)<\varepsilon$$

Hence

$$d(X_o^i\ X_o^j)\leq d(X_n^i\ X_n^j)+d(X_n^i\ X_o^i)+d(X_n^j\ X_o^j)<3\varepsilon$$

Thus $\{X_o^i\}$ is a Cauchy sequence in F^*. By the completeness of F^*, we know that there exists $X_o\in F^*$ such that for enough large i

$$d(X_o^i\ X_o)<\varepsilon$$

Therefore, for enough large n and fixed i

$$d(X_n\ X_o)\leq d(X_n^i\ X_n)+d(X_n^i\ X_o^i)+d(X_o^i\ X_o)<3\varepsilon$$

So $X=\{X_n\}$ is a convergent sequence and this proves the completeness of c.

Theorem 3.
c_o is a complete metric space with the metric f defined in the above theorems.

Proof. Be similar to the proof of theoreom 2.

Theorem 4.
l_p is a complete metric space with the metric h defined by

$$h(X,Y)=\left(\sum_n[d(X_n,Y_n)]^p\right)^{1/p}$$

Where $X=\{X_n\}$ and $Y=\{Y_n\}$ are sequences of F-numbers which are in l_p

Proof. Obviously $h(X,Y)\geq 0$. $h(X,Y)=0\Longleftrightarrow X=Y$ and $h(X,Y)=h(Y,X)$. The triangle inequality follows from Minkowski inequality and corresponding triangle inequality for d. Hence, (l_p,h) is a metric space.

To show the completeness of l_p, let $\{X^i\}$ be a Cauchy sequence in l_p. Then, for every fixed n, $\{X_n^i\}$ is a Cauchy sequence in F^*. Since (F^*,d) is complete, we have $\mathrm{Lim}_i X_n^i=X_n$ for each n. Put $X=\{X_n\}$, we now prove $\mathrm{Lim}_i X^i=X$ and $X\in l_p$.

Since $\{X^i\}$ is a Cauchy sequence in l_p, given $\varepsilon>0$ there exists an integer M such that for $i,j>M$

$$\sum_{n=1}^{\infty}[d(X_n^i,Y_n^j)]^p<\varepsilon$$

Hence, for every integer k

$$\sum_{n=1}^{k}[d(X_n^i\ X_n)]^p=\mathrm{Lim}_j\sum_{n=1}^{k}[d(X_n^i\ X_n^j)]^p\leq\varepsilon$$

Therefore

$$[h(X^i\ X)]^p = \sum_{n=1}^{\infty}[d(X_n^i\ X)]^p = \lim_k \sum_{n=1}^{k}[d(X_n^i\ X)]^p$$

$$= \overline{\lim_k}\ \lim_j \sum_{n=1}^{k}[d(X_n^i\ X_n^j)]^p \le \varepsilon$$

i.e. $\lim_i X^i = X$. From $h(X\ 0) \le h(X\ X^i) + h(X^i\ 0) < \infty$ we get $X \in l_p$.
The proof is completed.

Theorem 5.
s is a complete metric space with the metric g defined by

$$g(X,Y) = \sum_n \frac{1}{2^n} \cdot \frac{d(X_n\ Y_n)}{1 + d(X_n\ Y_n)}$$

Where $X=\{X_n\}$ and $Y=\{Y_n\}$ are arbitrary sequences of F-numbers.

Proof. Obviously $g(X,Y) \ge 0$. $g(X,Y)=0 \Leftrightarrow X=Y$ and $g(X,Y)=g(Y,X)$.
The triangle inequality $g(X,Y) \le g(X,Z) + g(Z,Y)$ follows from the function $\frac{t}{1+t}$ is monotonically increasing and corresponding inequality for d. Hence, s is a metric space with the metric g. To prove the completeness of s, let $\{X^i\}$ be a Cauchy sequence in s. Then given $\varepsilon > 0$, there exists an integer k such that fot $i,j>k$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d(X_n^i\ X_n^j)}{1 + d(X_n^i\ X_n^j)} < \varepsilon$$

It implies that $\{X_n^i\}$ is a Cauchy sequence in F^*. By the completeness of F^*, there exists X_n in F^* such that $\lim_i X_n^i = X_n$ for each n. Put $X=\{X_n\}$ we now prove that $\lim_i X^i = X$. Taking an integer m such that $\sum_{n=m}^{\infty} \frac{1}{2^n} < \varepsilon$
For each $n=1,2,\ldots,m-1$ there exists an integer M such that for $i>M$
$d(X_n^i\ X_n) < \varepsilon$.

Then

$$\sum_{n=1}^{m-1} \frac{1}{2^n} \frac{d(X_n^i\ X_n)}{1 + d(X_n^i\ X_n)} < \sum_{n=1}^{m-1} \frac{1}{2^n} \frac{\varepsilon}{1+\varepsilon} < \varepsilon$$

Therefore, for $i>M$

$$g(X^i\ X) = \left(\sum_{n=1}^{m-1} + \sum_{n=m}^{\infty}\right) \frac{1}{2^n} \cdot \frac{d(X_n^i\ X_n)}{1 + d(X_n^i\ X_n)} < \varepsilon + \varepsilon = 2\varepsilon$$

This proves the completeness of s.

Reference

[1] M.L.Puri and D.A.Ralescu, Fuzzy Random variables, J. Math. Anal. Appl. 114(1986) 409-422
[2] M.L.Puri and D.A.Ralescu, Differentials for Fuzzy Functions, J. Math. Anal. Appl. 91(1983) 552-558