About D.G. Schwartz's likelihood logic

Didier DUBOIS - Henri PRADE

Institut de Recherche en Informatique de Toulouse
Université Paul Sabatier, 118 route de Narbonne
31062 TOULOUSE Cedex - FRANCE

It is well-known (e.g. Dubois and Prade, 1988) that a measure of uncertainty defined on a Boolean algebra and taking its values in the interval [0,1] cannot be fully compositional with respect to all the logical connectives, just because we cannot equip [0,1] with a structure of Boolean algebra. Indeed probabilities are only compositional with respect to negation, i.e. $P(A) = 1 - P(\bar{A})$ (since $P(A \cup B) = P(A) + P(B)$ only when $A \cap B = \emptyset$, and $P(A \cap B) = P(A) \cdot P(B)$ requires the supplementary assumption of stochastic independence). Measures of possibility are only compositional with respect to disjunction, i.e. $\Pi(A \cup B) = \max(\Pi(A),\Pi(B))$ (and $\Pi(A \cap B) = \min(\Pi(A),\Pi(B))$ *only if* A and B are logically independent or if A = B), and necessity measures which are such that $N(A) = 1 - \Pi(\bar{A})$, are only compositional with respect to conjunction, i.e. $N(A \cap B) = \min(N(A),N(B))$.

Recently Schwartz (1992) has proposed a logic of likelihood governed by the following laws

$$\forall A \subseteq \Omega, \ \ell(\bar{A}) = 1 - \ell(A);$$

$$\forall A \subseteq \Omega, \forall B \subseteq \Omega, \ \ell(A \cup B) = \begin{cases} 1 \text{ if } A \cup B = \Omega \\ \max(\ell(A),\ell(B)) \text{ if not;} \end{cases}$$

$$\ell(A \cap B) = \begin{cases} 0 \text{ if } A \cap B = \emptyset \\ \min(\ell(A),\ell(B)) \text{ if not.} \end{cases}$$

As it can be seen such a measure of likelihood ℓ is as compositional as possible. Note that these likelihood set-functions are self-dual. Moreover only operations with a qualitative flavor are used to combine the likelihood degrees. Only a totally ordered set equipped with an order reversing involution is required as a likelihood scale. In the following we investigate what is the power of expressivity of these measures of likelihood, in the finite case.

Let $\Omega = \{\omega_1, \ldots, \omega_n\}$ be the finite set of atoms of the Boolean algebra 2^Ω. Let $\ell(\{\omega_i\}) = \ell_i \in [0,1]$. We have

$$\forall i, \ell_i = 1 - \ell(\Omega - \{\omega_i\}) = 1 - \max_{j \neq i} \ell_j = \min_{j \neq i} (1 - \ell_j).$$

If $\exists i, \ell_i = 1$ then $\ell(\Omega - \{\omega_i\}) = 0$ and then $\forall\, j \neq i, \ell_j = 0$. Thus it corresponds to the *deterministic* case.

Let us suppose that $\exists i, \ell_i = \alpha \in (0,1)$. Then

$$\forall\, j \neq i, \ell_j \leq \max_{k \neq i} \ell_k = \ell(\Omega - \{\omega_i\}) = 1 - \ell_i = 1 - \alpha.$$

Let us suppose that $\alpha = \ell_1 \geq \ell_2 \geq \ldots \geq \ell_n$. Then

$$\ell_2 = 1 - \ell(\Omega - \{\omega_2\}) = 1 - \max_{j \neq 2} \ell_j = 1 - \ell_1 = 1 - \alpha.$$

Since ℓ_1 is the maximal level, it follows that $\alpha \geq 1/2$. Similarly we have :

$$\ell_3 = 1 - \max(\ell_1, \ell_2, \ell_4, \ldots, \ell_n) = 1 - \alpha$$
$$\vdots$$
$$\ell_n = 1 - \alpha.$$

Thus if $\ell_1 < 1$, we can only have

$$1 > \ell_1 \geq \ell_2 = \ell_3 = \ldots = \ell_n > 0.$$

So we can only describe a *pseudo-deterministic* situation where $\exists i, \ell_i = \alpha \geq 1/2$, and $\forall\, j \neq i$, $\ell_j = 1 - \alpha \leq 1/2$. In particular, total uncertainty is described by $\forall i, \ell_i = \alpha = 1 - \alpha = 1/2$.

In this calculus, we only have four certainty levels corresponding respectively to the complete certainty of truth (1), the likelihood of truth (α), the unlikelihood of truth ($1 - \alpha$), and the complete certainty of falsity. Especially this representation of uncertainty does not really need the unit interval since only a 4-element totally ordered set $\{0, UL, L, 1\}$ is needed.

Thus this proposal corresponds to the most elementary logic of likelihood which can be imagined : there exists *one* alternative which, without being necessarily completely certain, appears to be more likely than the others which are considered as having a smaller, undifferentiated level of likelihood.

It is interesting to see whether likelihood measures induce a comparative probability ordering on events. Namely a comparative probability ordering $\geq$ is such that $\geq$ is complete and transitive, $A \geq \emptyset$, $\forall A \subseteq \Omega$, and $\geq$ satisfies the additivity axiom (Fine, 1973) :

$$\forall A, A \cap (B \cup C) = \emptyset, B > C \Leftrightarrow A \cup B > A \cup C \tag{1}$$

where $A > B$ means $A \geq B$ and not $(B \geq A)$. Any function ℓ classifies the events in Ω into 4 classes of level 1, L, UL and 0 respectively. Namely $\exists \omega_0$ such that the class of level L is $\{A \neq \Omega, \omega_0 \in A\}$, the class of level UL is $\{A \neq \emptyset, \omega_0 \notin A\}$. The class of level 1 is $\{\Omega\}$ and the one of level 0 is $\{\emptyset\}$. Particularly we have, for $A \neq B$,

$$A > B \text{ if and only if } A = \Omega \text{ or } B = \emptyset \text{ or } (\omega_0 \in A \text{ and } \omega_0 \notin B).$$

Let us consider whether (1) holds :

-) if $B = \Omega$ then $A = \emptyset$ and (1) is trivial. From now on $A \neq \emptyset$;

-) if $B \neq \Omega$, $C \neq \emptyset$ then assume $B > C$, i.e. $\omega_0 \in B$, $\omega_0 \notin C$. Since $A \cap B = \emptyset$, $\omega_0 \notin A$. Hence $\omega_0 \notin A \cup C$ and $A \cup B > A \cup C$.
Conversely assume $\Omega \neq A \cup B > A \cup C$. Clearly $A \cup C \neq \emptyset$; we have $\omega_0 \in A \cup B$, $\omega_0 \notin A \cup C$. Hence $\omega_0 \notin A$, and $\omega_0 \in B - C$. Hence $B > C$.
Assume now $A \cup B = \Omega > A \cup C$ then since $A \cap (B \cup C) = \emptyset$, it follows that $C \subseteq B$. If $\omega_0 \in B - C$ then $\ell(A \cup B) = 1 > \ell(A \cup C) = UL$ and $\ell(B) = L > \ell(C) = UL$. If $\omega_0 \in C$ we have $\ell(B) = \ell(C) = L$ and $\ell(A \cup B) = 1 > \ell(A \cup C) = L$. Hence (1) fails when $A \cup B = \Omega$.

-) when $C = \emptyset$ then (1) fails too, if $\omega_0 \in A$ since then $\ell(B) > \ell(C)$ but $\ell(A \cup B) = \ell(A \cup C) = L$, generally.

As a consequence the likelihood measure *almost* satisfies the axioms of a comparative probability relation. It satisfies the following reasonable relaxation of additivity :

$$\forall A, B, C \text{ such that } A \cup B \neq \Omega, A \cap (B \cup C) = \emptyset, C \neq \emptyset$$
$$B > C \Leftrightarrow A \cup B > A \cup C.$$

This paper also gives an answer to the following question : how far can we go with a representation of uncertainty that tries to take advantage of truth-functionality as far as possible. It is shown here that, not only truth-functionality per se is not possible, but retaining this property as much as mathematical consistency allows it leads to a very crude, almost deterministic model of uncertainty.

References

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