O-THEORY AND ITS RELATION TO FUZZY SET THEORY

Milan Lehotský, Vladimír Olej, Ján Chmúrny, Department of Technical Cybernetics, Technical University, 031 19 Liptovský Mikuláš, Czecho-Slovakia

Basic concepts, operators of O-theory and its use for representation of uncertainty in inference and reasoning systems are described. In our paper the relation between description of uncertainty in Dempster-Shafer theory and O-theory on one side and in fuzzy set theory on the other side are studied.

1. Introduction

The available information needed for inference and reasoning systems is in general often uncertain, unprecise and vague. Several ways exist for solving of this problem. It is probability theory (PT) [1], possibility theory, fuzzy set theory (FST) [2], Dempster-Shafer theory (DST) [3] and O-theory (Operator theory) [4-10].

O-theory is the theory of uncertainty based on DST which renders new way for solving uncertainty problems in inference and reasoning systems. It is a hybrid theory between FST, PT and DST. It was developed for representation and propagation of often occurring nonstandard forms of uncertain information and of information sources characterized by measurable undecidability and conflict.

2. Basic concepts of O-theory

Like in DST also in O-theory a set U called a universal set is considered and to each $x \in 2^U$, where 2^U is a power set of U, a nonnegative number m(x) called mass is assigned such that

$$\sum_{x \in 2^U} m(x) = 1 \qquad (1)$$

holds. Mass is a function

$$m: 2^U \to [0,1] \quad (2)$$

and is called mass distribution. In DST the following condition also holds $m(\emptyset) = 0$.

Various mass distributions can be defined on the same universal set. They will be denoted by $\underline{A}, \underline{B}$ and corresponding masses of $x \in 2^U$ $m_A(x)$ and $m_B(x)$. The mass of void set $m(\emptyset)$ can be nonzero in contrast to DST and so conflict can be represented.

For comparing of various sets from 2^U and of mass distributions the set cardinality denoted $|x|$ is used.

The dominance of set x over set $x' \in 2^U$ for mass distribution $\underline{A}$ is defined as follows:

$$x \succcurlyeq x', \text{ if } m_A(x)|x| \geq m_A(x')|x'| \quad (3)$$

The equality of two mass distributions is defined

$$\underline{A} = \underline{B} \Leftrightarrow m_A(x) = m_B(x) \text{ for each } x \in 2^U \quad (4)$$

The dominance of mass distribution $\underline{A}$ over $\underline{B}$ defined on the same set 2^U is defined as follows:

$$\underline{A} \succcurlyeq \underline{B}, \text{ if } \sum_{x \in 2^U} m_A(x)|x| \geq \sum_{x \in 2^U} m_B(x)|x| \quad (5)$$

The dominances are quasiorderings, ie. they are reflexive and transitive but they are not in general partial orderings (in contrast to the assertions in [4-8]) as the following examples show:

Example 1: Let $U=\{a,b\}$ and for mass distribution $\underline{A}$ $m_A(\{a\})= m_A(\{b\})=1/2$ holds. Then $m_A(\{a\})|\{a\}|=m_A(\{b\})|\{b\}|=1/2$ although $\{a\}=\{b\}$.

Example 2: Let $U=\{a,b\}$ and for mass distributions $\underline{A},\underline{B}$ $m_A(\{a\})=m_B(\{b\})=1$ holds. Then

$$\sum_{x\in 2^U} m_A(x)|x| = 1 \quad \text{and} \quad \sum_{x\in 2^U} m_B(x)|x| = 1 \qquad (6)$$

and so $\underline{A} \geqslant \underline{B}$ and $\underline{A} \leqslant \underline{B}$, while $\underline{A} \neq \underline{B}$.

3. Operators on mass distributions

For combining uncertainty distributions from various sources operators are needed which to one or more uncertainty distributions assign another distribution. In DST and in O-theory such an operator based on Dempster rule of combination is an intersection of two distributions $\underline{A},\underline{B}$ on the same universal set U $\underline{C} = \underline{A} \circledwedge \underline{B}$ defined as follows:

$$m_C(c) = \sum_{a\cap b=c} m_A(a)m_B(b) \qquad \text{for } a,b,c\in 2^U \qquad (7)$$

Another operator which has no counterpart in DST is a union of two distributions $\underline{A},\underline{B}$ on $\underline{C} = \underline{A} \circledvee \underline{B}$ defined as follows:

$$m_C(c) = \sum_{a\cup b=c} m_A(a)m_B(b) \qquad \text{for } a,b,c\in 2^U \qquad (8)$$

These operators are the counterparts of MIN and MAX used in FST. They are both commutative and associative but they are not distributive and idempotent. Intersection diminishes mass of sets with more elements and inversely increases mass of sets with few elements. Union has an inverse effect as the following example shows.

Example 3: Let $U=\{a,b\}$ and mass distributions $\underline{A}$ and $\underline{B}$ are given as follows:

$m_A(\emptyset) = 0.4$, $m_A(\{a\}) = 0.2$, $m_A(\{b\}) = 0.3$, $m_A(U) = 0.1$

$m_B(\emptyset) = 0.1$, $m_B(\{a\}) = 0.2$, $m_B(\{b\}) = 0.3$, $m_B(U) = 0.4$

Mass distributions for various operators are shown in the following table 1.

Table 1.

$x \in 2^U$	$\underline{A} ⓐ \underline{B}$	$\underline{A} ⓐ \underline{A}$	$\underline{B} ⓐ \underline{B}$	$\underline{A} ⓥ \underline{B}$	$\underline{A} ⓥ \underline{A}$	$\underline{B} ⓥ \underline{B}$
∅	0.58	0.76	0.31	0.04	0.16	0.01
{a}	0.14	0.08	0.20	0.14	0.20	0.08
{b}	0.24	0.15	0.33	0.24	0.33	0.15
U	0.04	0.01	0.16	0.58	0.31	0.76

The last basic operator is a complement. The complement $\tilde{\underline{A}}$ of a mass ditribution $\underline{A}$ is defined as follows:

$$m_{\tilde{A}}(x) = m_A(\bar{x}) \tag{9}$$

where $x \in 2^U$ and x is a set-theoretical complement of $x \in 2^U$. For these operators De Morgan's laws and involution

$$\widetilde{\underline{A} ⓥ \underline{B}} = \tilde{\underline{A}} ⓐ \tilde{\underline{B}}, \; \widetilde{\underline{A} ⓐ \underline{B}} = \tilde{\underline{A}} ⓥ \tilde{\underline{B}}, \; \tilde{\tilde{\underline{A}}} = \underline{A} \tag{10}$$

hold.

4. From mass assignments to membership functions

When we deal with the uncertain information on a universal set U, we have several possibilities to express it. Two of them - membership functions and mass assignments - we have mentioned in the previous part. Now a natural question arises - what are and what can be the relations between various expressions of uncertainty - membership functions and mass assignments.

Mass assignment gives more information as membership function because membership function expresses only uncertainty concerning the single elements while mass assignment gives the information about subsets of elements of the universal set and also about relations between various elements. Moreover, to each mass assigment a membership function on a given universal set U can be uniquely assigned by the following way [6]:

$$f(x) = \sum_{x \in X} m(X) \qquad (11)$$

The value of membership function in element x is also obtained by adding of the masses of all subsets containing element x. From the condition (1) it is clear that the function f is well defined in the sense that its values are in the closed interval [0,1].

A membership function can be after (11) uniquely assigned to the given mass assignment. In the following we will solve the problem how to find a mass assignment to a given membership function such that (11) holds.

5. Voting model

The problem given at the end of the previous part has in general infinite possible solutions. In practice it is important to find a meaningfull solution which can be precisely algorithmically described. One of these solution is the so--called voting model described in [9]. This model can be expressed precisely as follows:

Let $U=\{a_1, a_2, \ldots, a_n\}$ ordered in such way that for given membership function f

$$1 \geq f(a_1) \geq f(a_2) \geq \ldots \geq f(a_n) \geq 0 \qquad (12)$$

holds. Then we can construct following mass assignment on U:

$$\begin{aligned} m(\{a_1, \ldots, a_n\}) &= f(a_n) \\ m(\{a_1, \ldots, a_{n-1}\}) &= f(a_{n-1}) - f(a_n) \\ &\ \ldots \\ m(\{a_1, \ldots, a_k\}) &= f(a_k) - f(a_{k+1}) \\ &\ \ldots \\ m(\{a_1\}) &= f(a_1) - f(a_2) \\ m(\emptyset) &= 1 - f(a_1) \end{aligned}$$

and the masses of all other sets are equal to 0. It is clear that the function m is a mass assignment because the conditions (1) and (2) hold and that the condition (11) also holds such that the mass m is a given solution of our problem.

6. Mass assignment on two-element set

In the case of the two-element universal set U it is possible to describe all masses which can be assigned to the given mass distribution such that (11) holds.

Let $U=\{a,b\}$, f be a membership function on U such that $1\geq f(a)\geq f(b)\geq 0$. Denote $m_0 = m(\emptyset)$, $m_a = m(\{a\})$, $m_b = m(\{b\})$, $m_U = m(U)$. Then from (1) and (11) the following three equalities must hold:

$$\begin{aligned} f(a) &= m_a + m_U \\ f(b) &= m_b + m_U \\ 1 &= m_0 + m_a + m_b + m_U \end{aligned} \tag{13}$$

System (13) is the system of three equations with four unknowns which has an infinite number of solutions having the following description:

$$\begin{aligned} m_a &= f(a) - t \\ m_b &= f(b) - t \\ m_U &= t \\ m_0 &= 1 - f(a) - f(b) + t \end{aligned} \tag{14}$$

where t is a parameter.

We see that the conditions (1) and (11) hold. From the condition (2) (that means that all masses are values between 0 and 1) we can simply compute that for parameter t the following inequality holds:

$$\max\{f(a)+f(b)-1,0\} \leq t \leq \min\{f(a),f(b)\} \tag{15}$$

which can be equivalently expressed as

$$f(a) \otimes f(b) \leq t \leq f(a) \wedge f(b) \tag{16}$$

So we can conclude that for two-element set U all masses which can be assigned to the given membership function can be expressed by (14) where parameter t fulfils the condition (15) or (16).

7. Mass assignment on general finite set

In the case when the universal set has more than two elements it is not possible to express the mass assignment in such simple way as in the previous part because the system of linear equations analogical to (13) has too many unknowns. Instead of it we propose the following recursive description of mass assignment for a given membership function:

Let $U = \{a_1, a_2, \ldots, a_n\}$ and f is a given membership function on U such that

$$0 \le f(a_1) \le f(a_2) \le \ldots \le f(a_n) \le 1 \qquad (17)$$

holds. Then we take the sublattice of the lattice 2^U which contains all subsets containing element a_1 (see figure 1 for n=3). This lattice is isomorphic to the lattice $2^{U-\{a_1\}}$ of all subsets of U which do not contain the element a_1. We can consider a function m defined on this set for which (2) holds and instead (1)

$$\sum_{X \subset U-\{a_1\}} m(X) = f(a_1) \qquad (18)$$

holds.

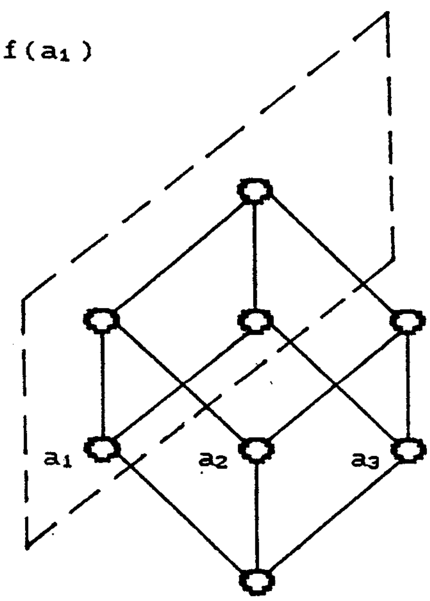

Figure 1.

The problem now is to find such a function m. We can now assign values to the subsets of U-{a_1} in such way that (2) and (18) hold. These values are now subtracted from the values of membership function f for elements $a_2,\dots,a_n$ in such way that from each f(a_i) is subtracted the sum of all m(X), where $a_i \in X$. The new membership fuction f' on U-{a_1} is obtained. From (17) it follows that all values f' are nonnegative. Now in the lattice U-{a_1} all elements are rearranged to fulfil the inequalities

$$0 \le f'(a_2') \le \dots \le f'(a_n') \le 1 \qquad (19)$$

This process will be repeated until the one-element set is obtained. During this process the sum of all values m must be controlled. If this sum exceeds one we must return to the previous lattice and change the values of m (it is always possible as the voting model shows). If at the end this sum is less than 1, the difference between 1 and this value is assigned to the void set and the definition of required mass assigment is completed.

8. Conclusion

O-theory can be used for design of computer circuits based on multivalued logics and in creation of inference and reasoning systems with uncertainty.

In this contribution mass assignments which can be constructed from a given membership function such that (11) holds.

References

[1] FELLER,W.: An Introduction to Probability Theory and Its Applications. John Wiley, New York, 1971.

[2] ZADEH,L.A.: Fuzzy Sets. Information and Control, Vol.8, 1965, No3, pp.338-353.

[3] CHATALIC,P.-DUBOIS,D.-PRADE,H.: An Approach to Approximate Reasoning Based on the Dempster Rule of Combination. International Journal of Expert Systems, Research and

Applications, Vol.1, 1987, No1, London, pp.67-85.

[4] OBLOW,E.M.: O-Theory - a Hybrid Uncertainty Theory. International Journal General Systems, Vol.13, 1987, pp. 95-106.

[5] OBLOW,E.M.: A Probabilistic - Propositional Framework for the O-Theory Intersection Rule. International Journal General Systems, Vol.13, 1987, pp.187-201.

[6] OBLOW,E.M.: Foundations of O-Theory: Measurements and Relation to Fuzzy Set Theory. International Journal General Systems, Vol.14, 1988, pp.357-378.

[7] OBLOW,E.M.: O-Theory: A Probabilistic Alternative to Fuzzy Set Theory. Uncertain Knowledge Based Systems. Pergamon Press, 1988, pp.111-119.

[8] OBLOW,E.M.: Extension of O-Theory to Problems of Logical Inferencing. Oak Ridge National Laboratory, Tennesee, 1989, 39p.

[9] BALDWIN,J.F.: Inference under Uncertainty for Expert Systems. ITRC 155, Research Report, University of Bristol, 1990, 13p.

[10] LEHOTSKÝ,M.-OLEJ,V.-CHMÚRNY,J,: Use of O-theory in Logical Inference Systems. Proceedings of Conf. Control of Discrete Processes, Bratislava, 1991, pp.68-69.