# BASIC THEOREMS OF TOLL SETS

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## ABSTRACT

In this paper, some basic properties, resolving theorem, expression theorem and extension principle are given about toll sets.

Keywords: Toll Sets, Indicator Funcion

Dubois D & Prade H have first proposed the concept of "toll set" in [1], and "toll measures" and "toll logic" are discussed. This paper, some important conclusions about toll sets are given.

## 1. DEFINITIONS AND BASIC PROPERTIES

**Definition 1.** The mapping

$$f:\ X:\to[0,+\infty]$$

is called a toll set on X. Set

$$T(X)=\{f \mid f:X\to[0,+\infty]\}$$

**Definition 2.** Let A be a subset of X. The mapping

$$I_A:\ X\to[0,+\infty]$$

$$x\mapsto I_A(x)=\begin{cases}0, & x\in A,\\ +\infty, & x\notin A.\end{cases}$$

is called an indicator function ( abbreviated to I−function) $I_A$ of A.

Clearly, the toll set is an I-function.

Definition 3. Let $*: [0,+\infty]^2 \to [0,+\infty]$, and $\forall a,b \in [0,+\infty]$, $\max\{a,b\} \leq a*b \leq a+b$. $\forall A, B, A_i$ ($i \in I$, I is a set of index) $\in T(X)$, $A \cup B$, $\bigcup_{i \in I} A_i$ and $A \subset B$ is definited as the follows: $\forall x \in X$,

$$(A \cup B)(x) = \min\{A(x), B(x)\} \qquad (\bigcup_{i \in I} A_i)(x) = \inf_{i \in I} A_i(x)$$

$$(A \cap B)(x) = A(x)*B(x).$$

If $C: T(X) \to T(X)$ and $\forall A \in T(X)$, $\forall x \in X$ $\min\{C(A)(x), A(x)\} = 0$, and $\max(C(A)(x), A(x)) > 0$, then C is called a complementary operation on $T(X)$, $C(A)$ is the complement of A, write it by $A^c$.

Definition 4. $\forall A, B \in T(X)$. A is contained in B, write it by $A \subset B$, if $\forall x \in X$, $A(x) \geq B(x)$

Theorem 1. $\forall A, B, C \in T(X)$

(1) $A \cap B \subset A \subset A \cup B$, $A \cap B \subset B \subset A \cup B$

(2) $A \cup B = B \cup A$

(3) $(A \cup B) \cup C = A \cup (B \cup C)$

(4) $A \cup (A \cap B) = A$, $A \cap (A \cup B) \subset A$

(5) $A \cup A = A$, $A \cap A \subset A$

(6) $X \cap A = A$, $X \cup A = X$, $\Phi \cap A = \Phi$, $\Phi \cup A = A$

(7) $A \cup A^c = X$

Where, $\forall x \in X$. $X(x) = 0$, $\Phi(x) = +\infty$

If

$$f: [0,+\infty] \to [0,1]$$
$$\beta \mapsto e^{-\beta} \qquad \text{(definite } e^{-(+\infty)} = 0)$$

then f is bijective and $\forall A, B \in T(X)$, $\forall x \in X$

$$f((A \cup B)(x)) = \max\{f((A)(x)), f((B)(x))\}$$

## 2. RESOLVING THEOREM OF TOLL SETS

*Let $A \in T(X)$, $a \in [0,+\infty]$,*

$$A_\alpha = \{x \mid x \in X,\ A(x) \leqslant \alpha\}$$

$$A_{\underset{\cdot}{\alpha}} = \{x \mid x \in X,\ A(x) < \alpha\}$$

$$(\alpha A_\alpha)(x) = \max\{\alpha, A_\alpha(x)\}$$

Where, $A_\alpha(.)$ is the I-function of $A_\alpha$.

$$(\alpha A_{\underset{\cdot}{\alpha}})(x) = \max\{\alpha, A_{\underset{\cdot}{\alpha}}(x)\}$$

Where, $A_{\underset{\cdot}{\alpha}}(.)$ is the I-function of $A_{\underset{\cdot}{\alpha}}$. Then

$$A = \bigcup_{\alpha \in [0,+\infty]} \alpha A_\alpha = \bigcup_{\alpha \in [0,+\infty]} \alpha A_{\underset{\cdot}{\alpha}}$$

## 3. EXPRESSION THEOREM OF TOLL SETS

Let $A \in T(X)$ $\forall \alpha_1, \alpha_2 \in [0,+\infty]$, $A_{\underset{\cdot}{\alpha}1} \subset A_{\alpha 1} \subset A_{\alpha 2}$ and $A_{\underset{\cdot}{\alpha}1} \subset A_{\underset{\cdot}{\alpha}2}$, if $\alpha_1 \leqslant \alpha_2$.

If the mapping $H$: $[0,+\infty] \to P(X)$ is such that $\forall \alpha_1, \alpha_2 \in [0,+\infty]$, $H(\alpha_1) \subset H(\alpha_2)$ if $\alpha_1 \leqslant \alpha_2$, and set $A: X \to [0,+\infty]$ is that

$$A(x) = \Big(\bigcup_{\alpha \in [0,+\infty]} \alpha H(\alpha)\Big)(x)$$

Where, $H(\alpha)(.)$ is the I-function of $H(\alpha)$. Then

$$\forall \alpha \in [0,+\infty],\ A_\alpha = \bigcap_{\lambda > \alpha} H(\lambda),\ A_{\underset{\cdot}{\alpha}} = \bigcap_{\lambda < \alpha} H(\lambda)$$

## 4. EXTENSION PRINCIPLE OF TOLL SETS

Extension Principle. Let $f: X \to Y$. A new mapping is introduced by $f$, write it by $f$ again.

$$f:\ T(X) \to T(Y)$$

$$A \mapsto f(A)$$

$$\forall y \in Y,\ (f(A))(y) = \begin{cases} \inf\{A(x) \mid x \in f^{-1}(y)\}, & f^{-1}(y) \neq \Phi \\ +\infty & \text{otherwise,} \end{cases}$$

The other mapping introduced by f is $f^{-1}$.

$$f^{-1}: T(Y) \to T(X)$$
$$B \mapsto f^{-1}(B)$$
$$\forall x \in X, \ (f^{-1}(B))(x) = B(f(x))$$

Theorem 2. Let $f: X \to Y$.

(1) $f(A) = \Phi$ iff $A = \Phi$

(2) $f(X) = I_{f(x)}$

(3) $A \subset B \Rightarrow f(A) \subset f(B)$

(4) $f(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f(A_i)$

(5) $f^{-1}(\Phi) = \Phi$. $f^{-1}(B) = \Phi$ iff $B = \Phi$, if f is surjective.

(6) $f^{-1}(Y) = X$

(7) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$

(8) $f^{-1}(\bigcup_{i \in I} B_i) = \bigcup_{i \in I} f^{-1}(B_i)$

(9) $A \subset (f^{-1} \circ f)(A)$. $A = (f^{-1} \circ f)(A)$ if f is injective.

(10) $(f \circ f-1)(B) \subset B$. $(f \circ f^{-1})(B) = B$ if f is surjective.

Theorem 3. Let $f: X \to Y$, $A \in T(X)$. Then

$$f(A) = \bigcup_{\alpha \in [0, +\infty]} \alpha f(A_\alpha)$$

## REFERENCES


[1] Dubois D & Prade H, Toll Sets, Proc. of the 4th IFSA Congress—Artificial Intelligence, (July, 1991, Brussels), 21—24.

[2] Zhang Wenxiou, Wang Guojun, Liou Wangjin & Fang Jinxuan. Introduction to Fuzzy Mathematics, Xian Jiaotong University Publishing House (in Chinese), 1991