FROM FUZZY SUBALGEBRAS TO FUZZY SUBGROUPS

by László Filep

Department of Mathematics, Bessenyei College,

Nyiregyháza, H-4401, Pf. 166, Hungary

Abstract: Here we use a new approach to define fuzzy subalgebras of an algebra, then we define the fuzzy subgroups of a (2,1)-type algebra being not necessarily a group.

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I. Fuzzy subalgebras of an algebra

Let $A=(A,F)$ be an algebra with finite set F of finitary operations, where F contains at most one nullary operation, i.e. at most one constant being also the identity of each operation. Further, let $P=(P,G,\leq)$ be ordered algebra of the same type. The order be isotone with each operation in all variables. Namely, for the constant c, let $c\geq p$ for all $p\in P$.

A map $\mu: A\longrightarrow P$ is called a P-fuzzy subset of A, and their family is denoted by A^P. A $\mu\in A^P$ is said to be a P-fuzzy subalgebra of the algebra A, if in the following diagram

$$\begin{array}{ccc} A^n & \xrightarrow{\pi\mu^n} & P^n \\ f\downarrow & & \downarrow g \\ A & \xrightarrow{\mu} & P \end{array} \qquad \begin{array}{l} \pi\mu^n(a_1,\ldots,a_n)= \\ =(\mu(a_1),\ldots,\mu(a_n)), \\ \forall\ a_1,\ldots,a_n\in A. \end{array}$$

$\mu\cdot f\geq g\cdot\pi\mu^n$ holds, where $f\in F$ and $g\in G$ are corresponding n-ary operations.

In details:

$\mu(fa_1\ldots a_n) \geq g(\mu(a_1),\ldots,\mu(a_n)) = g\mu(a_1)\ldots\mu(a_n)$, $n\geq 1$;

$\mu(f\emptyset) = k \geq g\emptyset = c$, $n = 0$.

By the type of a fuzzy subalgebra we mean the type of the basic algebra A.

To show that this definition, which is different from the usual ones, is natural, we have to check its connection with the corresponding crisp notion through its cuts. Here we shall use a generalized cut instead of the usual α-cut.

The subset Q of an ordered set $(P,\leq)$ having the property

$$(p \in P \wedge q \in Q \wedge p \geq q) \Longrightarrow p \in Q$$

is called a right segment of P. If $Q\subseteq P$ is a right segment of P and $\mu\in A^{P}$, then the Q-cut μ_{Q} of μ is defined by

$$\mu_{Q} = \{a \mid a \in A,\ \mu(a) \in Q\}$$

In the followings A always denotes an algebra defined above, and P an ordered algebra of the same type.

Theorem 1.1. If each non-empty Q-cut μ_{Q} of a $\mu\in A^{P}$ is a subalgebra of A, then μ is a P-fuzzy subalgebra of A.

Proof. Let A have a constant k. Since k must be contained in each subalgebra of A, hence $\mu(k) \geq \mu(a)$ must hold for all $a\in A$. Otherwise, $\mu(k) < \mu(a)$ for some $a\in A$ would mean that choosing

$$Q = \{q \mid q \in P,\ \mu(q) \geq \mu(a)\}$$

we would get $k \notin \mu_{Q} \neq \emptyset$, which contradicts our assumption.

Now, let $a_{1},\dots,a_{n} (n\geq1)$ be arbitrary elements of A, and let $f\in F$ an n-ary operation $(n\geq1)$. Choose the right segment Q of P, the smallest element of which is $g\mu(a_{1})\dots\mu(a_{n})$, where $g\in G$ is the corresponding n-ary operation. From the isotonity we get

$$g\mu(a_{1})\dots\mu(a_{n}) \leq g\,k\dots k\mu(a_{i})k\dots k \leq \mu(a_{i}),$$

that is $a_{i}\in\mu_{Q}$ for all $i=1,\dots,n$. Being μ_{Q} a subalgebra of A, we get from this that

$$fa_{1}\dots a_{n} \in \mu_{Q},$$

too, that is

$$\mu(fa_{1}\dots a_{n}) \geq g\mu(a_{1})\dots\mu(a_{n}),$$

which was to be proved.

Theorem 1.2. If $\mu\in A^{P}$ is a P-fuzzy subalgebra of A, and the right segment $Q\subseteq P$ is a subalgebra of P, then any non-empty Q-cut μ_{Q} of μ is a subalgebra of A.

Proof. Since $\mu(k)\geq c$ and $c\geq p$ for all $p\in P$ by definition, hence $\mu(k) \geq \mu(a)$ for all $a\in A$. This means that $k\in\mu_{Q}\neq\emptyset$ in any case. Thus μ_{Q} is closed under the nullary operation.

Let μ_{Q} be a non-emty Q-cut of μ, where Q is a subalgebra of P. If $a_{1},\dots,a_{n}\in \mu_{Q}$, that is $\mu(a_{1}),\dots,\mu(a_{n}) \in Q$, then being Q a subalgebra we have

$$g\mu(a_{1})\dots\mu(a_{n}) \in Q$$

for the corresponding $g\in G$ n-ary operation.

Further, since μ is a P-fuzzy subalgebra, therefore

$$\mu(fa_1 \ldots a_n) \in Q,$$

consequently

$$fa_1 \ldots a_n \in \mu_Q$$

which proves that μ_Q is a subalgebra of A.

2. Fuzzy subgroups of a (2,1)-type algebra

Let A be a (2,1)-type algebra, say $A = (A, \cdot, ^{-1})$. Further, let P be also a (2,1)-type ordered algebra, say $P = (P, *, ^{-})$. Then a P-fuzzy subalgebra $\mu \in A^P$ of A is a P-fuzzy subgroup of A (shortly a fuzzy subgroup), if for all $a, b, c \in A$

(i) $\mu((ab)c) = \mu(a(bc))$,

(ii) $\mu((ab)b^{-1}) = \mu(b^{-1}(ba)) = \mu(a)$.

If additionally, for all $a, b \in A$

(iii) $\mu(ab) = \mu(ba)$

then μ is called a fuzzy Abelian group.

It is easily seen that if A is a group, then (i) - the fuzzy associativity - and (ii) trivially hold. Moreover, if $P = [0,1]$, $*$ = min, $-$ = identical mapping, then we get the classical concept of fuzzy group introduced by A. Rosenfeld in 1971. For these fuzzy groups (iii) was used later to define fuzzy normal subgroups.

From now on P is replaced by a meet semilattice L, where the identical mapping will be the unary operation.

Theorem 2.1. Let $\mu \in L^A$ be fuzzy subgroup, where L is totally ordered. Then $\mu(a) \neq \mu(b)$ implies $\mu(ba) = \mu(b) \wedge \mu(a)$ for all $a, b \in A$.

Proof. Let e.g. $\mu(b) > \mu(a)$. For $\mu(ba)$ we have two cases:

1. $\mu(ba) \geq \mu(b)$. Then

$$\mu(a) = \mu(b^{-1}(ba)) \geq \mu(b^{-1}) \wedge \mu(ba) = \mu(b) \wedge \mu(ba) = \mu(b)$$

consequently $\mu(a) \geq \mu(b)$, which is a contradiction.

2. $\mu(ba) \leq \mu(b)$. Then

$$\mu(a) = \mu(b^{-1}(ba)) \geq \mu(b^{-1}) \wedge \mu(ba) = \mu(b) \wedge \mu(ba) =$$
$$\mu(ba) \geq \mu(b) \wedge \mu(a) = \mu(a),$$

consequently

$$\mu(ba) = \mu(b) \wedge \mu(a),$$

as we stated.

Lemma 2.1. Let $\mu \in L^A$ be a fuzzy subgroup.

Then $\mu(a^{-1}a) \geq \mu(a), \mu(aa^{-1}) \geq \mu(a)$ for all $a \in A$.

Proof. Clear.

Theorem 2.2. Let $\mu \in L^A$ be a fuzzy subgroup, where L is totally ordered. Then for all $a, b \in A$

(i) $\mu(b^{-1}b) \geq \mu(a), \mu(bb^{-1}) \geq \mu(b)$;

(ii) $\mu(b^{-1}b) = \mu(a^{-1}a), \mu(bb^{-1}) = \mu(aa^{-1})$.

Proof. (i) By definition

$$\mu(a) = \mu(b^{-1}(ba)) = \mu((b^{-1}b)a)$$

Now we have two cases:

1. $\mu(b^{-1}) = \mu(a)$. The statement is true.
2. $\mu(b^{-1}b) \neq \mu(a)$. Then by Theorem 1:

$$\mu(a) = \mu((b^{-1}b)a) = \mu(b^{-1}b) \wedge \mu(a),$$

consequently $\mu(b^{-1}b) \geq \mu(a)$. With this we have proved $\mu(b^{-1}b) \geq \mu(a)$ in any case. The proof of the inequality $\mu(bb^{-1}) \geq \mu(a)$ is similar.

(ii) Let $\mu(a^{-1}a) = c_1, \mu(b^{-1}b) = c_2$. Suppose $c_1 \neq c_2$, say $c_1 > c_2$. Then by (i) $c_2 = \mu(b^{-1}b) \geq \mu(d)$ for all $d \in A$, specially for $d = a^{-1}a$. That is $c_2 = \mu(b^{-1}b) \geq \mu(a^{-1}a) = c_1$ which contradicts $c_1 > c_2$.

Lemma 2.2. Let $\mu \in L^A$ be a fuzzy supgroup, and denote $b^{-1}b = e$ for some $a \in A$. Then $\mu(a) = \mu(ea)$ for all $a \in A$.

Proof. $\mu(a) = \mu(b^{-1}(ba)) = \mu((b^{-1}b)a) = \mu(ea)$.

Lemma 2.3. Let $\mu \in L^A$ be a fuzzy subgroup, where L is totally ordered. If $\mu(x) \neq \mu(y)$ for all $x, y \in A$, then $a^{-1}a = e$ for all $a \in A$.

Proof. Suppose that $a^{-1}a = e$, $b^{-1}b = f$, and $e \neq f$ for some $a, b \in A$. Then by Theorem 2.2

$$\mu(e) = \mu(a^{-1}a) = \mu(b^{-1}b) = \mu(f),$$

which is a contradiction.

References:

1. P.S.Das, Fuzzy groups and level subgroups. Journal Math.Anal. Appl., 84/1981/, 264-269.
2. L.Filep, Fundamentals of a general theory of fuzzy relations and algebras. Proc.fourth IFSA World Congress. Brussels, 1991. pp. 70-73.