FUZZY ALGEBRAIC VARIETIES

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Abstract: The concept of a fuzzy algebraic variety is introduced in order to bring the current knowledge of fuzzy commutative ring theory to bear on the solution of nonlinear systems of equations of fuzzy singletons. It is shown for every finite-valued fuzzy ideal A of a polynomial ring in several indeterminates over a field with $A(0) = 1$ that the fuzzy algebraic variety of A can be expressed as a union of fuzzy irreducible algebraic varieties, no one of which is contained in the union of the others.

INTRODUCTION

Rosenfeld's application [12] of the pioneering work of Zadeh [15] inspired the fuzzification of various algebraic structures. Liu [2] and Mukherjee and Sen [10] presented some of the earliest work on the fuzzification of an ideal of a ring. Since then the notions of fuzzy prime ideal, fuzzy primary ideal, the radical of a fuzzy ideal, and the fuzzy primary representation of a fuzzy ideal have been introduced and examined, [1,3,4,5,6,7,10,11,13,14,16,17,18,19]. There are several natural ways to define these concepts, many of which have appeared in the literature. In the interesting work of Zadehi [16] and Kumbhojkar and Bapat [1], the various types of fuzzy prime ideals, fuzzy primary ideals, the radical of a fuzzy ideal have been compared. In [7], various types of radicals of fuzzy ideals and fuzzy primary representations of fuzzy ideals have been compared in order to prepare the way for the study of nonlinear systems of equations of fuzzy singletons.

Up to this point, the fuzzification of concepts and results of commutative ring theory have had no apparent application. The purpose of this paper is to give some meaning to fuzzy commutative ring theory developed to this point and to put some direction to its further study. We bring fuzzy commutative ring theory to bear on a natural application area, namely, the solution of nonlinear systems of equations of fuzzy singletons. Let R denote the polynomial ring $F[x_1, ..., x_n]$ where F is a field and $x_1, ..., x_n$ are algebraically independent indeterminants over F. Let L be a field containing F. L may be taken to be the algebraic closure of F or an algebraically closed field with infinite transcendence degree over F. Let L^k denote the set of all ordered k-tuples with entries from L, k a positive integer. Our approach is to consider those fuzzy ideals A of R which are finite-valued and are such that $A(0) = 1$ since these are precisely the fuzzy ideals of R which have fuzzy primary representations, [6]. We define the fuzzy algebraic variety $\mathcal{M}(A)$ of A and show that from an irredundant fuzzy primary representation of A, $\mathcal{M}(A)$ is a finite union of irreducible fuzzy algebraic varieties, no one of which is contained in the union of the others, Theorem 2.7. We then apply this result to the solution of a nonlinear system of equations of fuzzy singletons, Example 2.9. We show that $\exists$ a fuzzy ideal A of R which represents this system and the irredundant primary representation of $\sqrt{A}$ displays the solution of the system in a manner similar to that of the crisp situation. Hence we have thus shown that in this sense the current definitions of fuzzy prime ideal, fuzzy primary ideal, and radical of a fuzzy ideal have been appropriately defined.

A fuzzy subset of a set Z is a function of Z into the closed interval [0, 1]. If X and Y are fuzzy subsets of Z, then we write $X \subseteq Y$ if $X(z) \leq Y(z)$ $\forall z \in Z$. If $\{X_i \mid i \in I\}$ is a collection of fuzzy subsets of Z, we define the fuzzy subsets $\bigcap_{i \in I} X_i$ and $\bigcup_{i \in I} X_i$ of Z by $\forall z \in Z$, $(\bigcap_{i \in I} X_i)(z) = \inf\{X_i(z) \mid i \in I\}$ and $(\bigcup_{i \in I} X_i)(z) = \sup\{X_i(z) \mid i \in I\}$. Let X be a fuzzy subset of Z. We let Im(X) denote the image of X and |Im(X)| the cardinality of Im(X). We say that X is finite-valued if $|Im(X)| < \infty$. We let $X^* = \{z \in Z \mid X(z) > 0\}$, the support of X, and $X_t = \{z \in Z \mid X(t) \geq t\}$ $\forall t \in [0, 1]$. If X is a fuzzy ideal of R [4, Lemma 1.7, p. 94], then X^* is an ideal of R. Also X is a fuzzy ideal of R if and only if X_t is an ideal of R $\forall t \in Im(X)$, [14, Theorem 1.2, p. 98]. For $z \in Z$ and $t \in [0, 1]$, we let z_t denote the fuzzy subset of Z defined by $z_t(z) = t$ and $z_t(x) = 0$ if $x \neq z$. z_t is called a fuzzy singleton of Z. If z_t and y_s are fuzzy singletons of R, we define $z_t + y_s = (z + y)_r$ and $z_t y_s = (zy)_r$ where $r = \min\{t, s\}$. If S is a subset (fuzzy subset) of R, we let <S> denote the ideal (fuzzy ideal) of R generated by S. We recall that a fuzzy subset A of R is a fuzzy ideal of R iff $\forall x, y \in R$, $A(x - y) \geq \min\{A(x), A(y)\}$ and $A(xy) \geq \max\{A(x), A(y)\}$. If A is a fuzzy ideal of R, then the radical of A, $\sqrt{A}$, is the intersection of all fuzzy prime ideals of R which contain A, [5, Definition 4.3, p. 244], [16, Theorem 3.8].

1. FUZZY ALGEBRAIC VARIETIES

If I is an ideal of R, we let $\mathcal{M}(I)$ denote the algebraic variety of I, [8, p. 203]. If Z is a subset of L^k, we let $\mathfrak{I}(Z)$ denote the set of all $f \in R$ which vanish at all points of Z. Then $\mathfrak{I}(Z)$ is an ideal of R, [8, p. 203]. We now give definitions for the fuzzy counterparts of $\mathcal{M}$ and $\mathfrak{I}$. Let c be a strictly decreasing function of [0, 1] into itself such that $c(0) = 1$, $c(1) = 0$, and $\forall t \in [0, 1]$, $c(c(t)) = t$. The following approach has the advantage that c may be changed to fit the application. For example, c may belong to the Sugeno class of fuzzy complements for one application and the Yeager class for another.

Definition 1.1. Let X be a finite-valued fuzzy subset of L^k, say $\mathrm{Im}(X) = \{t_0, t_1, ..., t_n\}$ where $t_0 < t_1 < ... < t_n$. Define the fuzzy subset $\mathfrak{I}(X)$ of R as follows:

$$\mathfrak{I}(X)(f) = \begin{cases} c(t_n) \text{ if } f \in R - \mathfrak{I}(X_{t_n}); \\ c(t_i) \text{ if } f \in \mathfrak{I}(X_{t_{i+1}}) - \mathfrak{I}(X_{t_i}), i = 1, ..., n-1; \\ c(t_0) \text{ if } f \in \mathfrak{I}(X_{t_1}). \end{cases}$$

If $n = 0$, then we define $\mathfrak{I}(X)(0) = 1$.

Definition 1.2. Let A be a finite-valued fuzzy ideal of R, say $\mathrm{Im}(A) = \{s_0, s_1, ..., s_m\}$ where $s_0 < s_1 < ... < s_m$. Define the fuzzy subset $\mathcal{M}(A)$ of L^k as follows:

$$\mathcal{M}(A)(b) = \begin{cases} c(s_m) \text{ if } b \in L^k - \mathcal{M}(A_{s_m}); \\ c(s_i) \text{ if } b \in \mathcal{M}(A_{s_{i+1}}) - \mathcal{M}(A_{s_i}), i = 1, ..., m-1; \\ c(s_0) \text{ if } b \in \mathcal{M}(A_{s_1}). \end{cases}$$

$\mathcal{M}(A)$ is called a **fuzzy algebraic variety** (of A).

In Definition1.1, it is possible for $\mathfrak{I}(X_{t_{i+1}}) = \mathfrak{I}(X_{t_i})$ or $R = \mathfrak{I}(X_{t_n})$. In this case $c(t_i) \notin \mathrm{Im}(\mathfrak{I}(X))$, $i = 1, ..., n$. Similarly, it is possible for $c(s_i) \notin \mathrm{Im}(\mathcal{M}(A))$ for some $i = 1, ..., m$.

Proposition 1.3. Let X be defined as in Definition 1.1. Then

(1) $\mathfrak{I}(X)_{c(t_i)} = \mathfrak{I}(X_{t_{i+1}})$ for $i = 0, 1, ..., n-1$;

(2) if $0 \leq s \leq c(t_n)$, then $\mathfrak{I}(X)_s = R$;

(3) if $c(t_{i+1}) < s < c(t_i)$, then $\mathfrak{I}(X)_s = \mathfrak{I}(X_{c(s)})$ for $i = 0, 1, ..., n-1$;

(4) if $c(t_0) < s \leq 1$, then $\mathfrak{I}(X)_s = \emptyset$.

For X as defined in Definition 1.1, $\mathfrak{I}(X)_{c(t_i)} = \mathfrak{I}(X_{t_{i+1}})$ is an ideal of R for $i = 0, 1, ..., n-1$. Thus since $\mathrm{Im}(X) \subseteq \{c(t_i) \mid i = 0, 1, ..., n\}$, $\mathfrak{I}(X)$ is a fuzzy ideal of R, [14, Theorem 1.2, p. 98].

Proposition 1.4. Let A be defined as in Definition 1.2. Then

(1) $\mathcal{M}(A)_{c(s_i)} = \mathcal{M}(A_{s_{i+1}})$ for $i = 0, 1, ..., m-1$;

(2) if $0 \leq t \leq c(s_m)$, then $\mathcal{M}(A)_t = L^k$;

(3) if $c(s_{i+1}) < t < c(s_i)$, then $\mathcal{M}(A)_t = \mathcal{M}(A_{c(t)})$ for $i = 0,1, ..., m-1$;

(4) if $c(s_0) < t \leq 1$, then $\mathcal{M}(A) = \emptyset$.

Proposition 1.5. Let X and A be as defined on Definitions 1.1 and 1.2, respectively. Then

(1) $|\mathrm{Im}(\mathcal{M}(\mathfrak{I}(X))| = |\mathrm{Im}(\mathfrak{I}(X))|$;

(2) $|\mathrm{Im}(\mathfrak{I}(\mathcal{M}(A))| = |\mathrm{Im}(\mathcal{M}(A))|$.

Proposition 1.6. Let X and A be defined as in Definitions 1.1 and 1.2, respectively. Then

(1) $\forall$ t $\in$ [0, 1], $\mathcal{M}(\mathfrak{I}(X))_t = \mathcal{M}(\mathfrak{I}(X_t))$;

(2) $\forall$ s $\in$ [0, 1], $\mathfrak{I}(\mathcal{M}(A))_s = \mathfrak{I}(\mathcal{M}(A_s))$.

Proposition 1.7. Let X and A be defined as in Definitions 1.1 and 1.2, respectively. Then

(1) $\mathfrak{I}(\mathcal{M}(\mathfrak{I}(X))) = \mathfrak{I}(X)$;

(2) $\mathcal{M}(\mathfrak{I}(\mathcal{M}(A))) = \mathcal{M}(A)$.

Theorem 1.8. Let M be a fuzzy subset of L^k. Then M is a fuzzy algebraic variety if and only if M is finite-valued and $\forall$ t $\in$ Im(M), M_t is an algebraic variety.

Proposition 1.9. If A is a nonconstant fuzzy prime ideal of R, then $A = \mathfrak{I}(\mathcal{M}(A))$.

Theorem 1.10. Suppose that A is defined as in Definition 1.2 and A(0) = 1. Then $\mathcal{M}(A) = \mathcal{M}(\sqrt{\bar{A}})$.

Corollary 1.11. If P is a fuzzy prime ideal of R belonging to the fuzzy primary ideal Q of R, then $\mathcal{M}(Q) = \mathcal{M}(P)$.

Lemma 1.12. Suppose that A and B are fuzzy ideals of R such that $Im(A) = \{s_0, s_1, ..., s_m\}$ where $s_0 < s_1 < ... < s_m = 1$ and Im(B) = {s, 1} where s < 1. Then $\mathcal{M}(A \cap B) = \mathcal{M}(A) \cup \mathcal{M}(B)$.

Theorem 1.13. If A and B are finite-valued fuzzy ideals of R such that A(0) = B(0) = 1, then $\mathcal{M}(A \cap B) = \mathcal{M}(A) \cup \mathcal{M}(B)$.

Lemma 1.14. Suppose that M and N are fuzzy algebraic varieties such that $Im(M) = \{t_0, t_1, ..., t_n\}$ where $0 = t_0 < t_1 < ... < t_n$ and Im(N) = {0, t} where 0 < t. Then $\mathfrak{I}(M \cup N) = \mathfrak{I}(M) \cap \mathfrak{I}(N)$.

Theorem 1.15. If M and N are fuzzy algebraic varieties such that 0 $\in$ Im(A) $\cap$ Im(B), then $\mathfrak{I}(M \cup N) = \mathfrak{I}(M) \cap \mathfrak{I}(N)$.

IRREDUCIBLE FUZZY ALGEBRAIC VARIETIES

Definition 2.1. Let M be a fuzzy algebraic variety. Then M is **irreducible** if and only if $\forall$ fuzzy algebraic varieties M' and M" such that M = M' $\cup$ M" either M = M' or M = M"; otherwise M is called **reducible**.

Theorem 2.2. Let M be a fuzzy algebraic variety. Then M is irreducible and nonconstant if and only if Im(M) = {0, t}, 0 < t, and M_t is irreducible.

Theorem 2.3. Let A be a nonconstant finite-valued fuzzy ideal of R. Then $\mathfrak{I}(\mathcal{M}(A))$ is prime if and only if $\mathcal{M}(A)$ is irreducible.

Theorem 2.4. Let A be a finite-valued fuzzy ideal of R with $A(0) = 1$. Then $\mathcal{I}(\mathcal{M}(A)) = \sqrt{A}$.

Corollary 2.5. Let A and B be finite-valued fuzzy ideals of R such that $A(0) = B(0) = 1$. Then $\mathcal{M}(A) \subset \mathcal{M}(B)$ if and only if $\sqrt{A} \supset \sqrt{B}$.

Theorem 2.6. There exists a one-to-one correspondence between fuzzy algebraic varieties M with $0 \in \text{Im}(M)$ and fuzzy radical ideals.

Theorem 2.7. Every fuzzy algebraic variety M with $0 \in \text{Im}(M)$ can be uniquely expressed as the union of a finite number of irreducible algebraic varieties no one of which is contained in the union of the others.

Given a finite-valued fuzzy ideal A of R with $A(0) = 1$. Let $\sqrt{A} = P_1 \cap \ldots \cap P_r$ be a fuzzy irredundant primary representation. Then the P_i are the minimal fuzzy prime ideals belonging to A, [6, Theorem 3.17, p. 161]. We thus may obtain $\mathcal{M}(A)$ as the union of fuzzy algebraic varieties of the minimal fuzzy prime ideals among the fuzzy prime ideals belonging to the fuzzy primary ideals in an irredundant primary representation of A.

Theorem 2.8. [9] Let S be a fuzzy subset of R. Then $\forall\, x \in R$, $\langle S\rangle(x) = \sup\{(\sum_{i=1}^{n}(r_i)_1(x_i)_{t_i})(x) \mid r_i, x_i \in R, t_i \le S(x_i), i = 1, \ldots, n; n \in N\}$ where N denotes the positive integers.

Example 2.9. Let $R = F[x, y, z]$ where F is the field of complex numbers and x, y, z are algebraically independent indeterminants over F. Define the fuzzy subset A of R by $A(0) = 1$, $A(f) = 1/2$ if $f \in \langle x^2z\rangle - \langle 0\rangle$, $A(f) = 1/4$ if $f \in \langle x^2 + y^2 - 1, x^2z\rangle - \langle x^2z\rangle$, and $A(f) = 0$ if $f \in R - \langle x^2 + y^2 - 1, x^2z\rangle$. Then A is a fuzzy ideal of R. Now $\sqrt{A}$ is such that $\sqrt{A}(0) = 1$, $\sqrt{A}(f) = 1/2$ if $f \in \langle xz\rangle - \langle 0\rangle$, $\sqrt{A}(f) = 1/4$ if $f \in \langle x^2 + y^2 - 1, xz\rangle - \langle xz\rangle$, and $\sqrt{A}(f) = 0$ if $f \in R - \langle x^2 + y^2 - 1, xz\rangle$. Hence

$A_0 = R$	$\sqrt{A_0} = R$
$A_{1/4} = \langle x^2 + y^2 - 1, x^2z\rangle$	$\sqrt{A_{1/4}} = \langle x^2 + y^2 - 1, xz\rangle$
$A_{1/2} = \langle x^2z\rangle$	$\sqrt{A_{1/2}} = \langle xz\rangle$
$A_1 = \langle 0\rangle$	$\sqrt{A_1} = \langle 0\rangle$

Since $F^k = \mathcal{M}(\langle 0\rangle)$,

$$\mathcal{M}(A)(b) = \begin{cases} c(1/2) \text{ if } b \in \mathcal{M}(\langle 0\rangle) - \mathcal{M}(\langle xz\rangle), \\ c(1/4) \text{ if } b \in \mathcal{M}(\langle xz\rangle) - \mathcal{M}(\langle x^2 + y^2 - 1, xz\rangle), \\ c(0) = 1 \text{ if } b \in \mathcal{M}(\langle x^2 + y^2 - 1, xz\rangle). \end{cases}$$

Consider the fuzzy subsets W, X, Y of R defined by $W(f) = 1$ if $f \in \langle x^2 + y^2 - 1, x^2 z\rangle$, $W(f) = 0$ otherwise; $X(f) = 1$ if $f \in \langle x^2z\rangle$, $X(f) = 1/4$ otherwise; $Y(f) = 1$ if $f \in \langle 0\rangle$, $Y(f) = 1/2$ otherwise.

Then W, X, Y are fuzzy ideals of R and $A = W \cap X \cap Y$. Define the fuzzy subsets $Q^{(i)}$ of R, i = 1, ..., 6 by $Q^{(1)}(f) = 1$ if $f \in \langle x^2, y - 1\rangle$, $Q^{(1)}(f) = 0$ otherwise; $Q^{(2)}(f) = 1$ if $f \in \langle x^2, y + 1\rangle$, $Q^{(2)}(f) = 0$ otherwise; $Q^{(3)}(f) = 1$ if $f \in \langle x^2 + y^2 - 1, z\rangle$, $Q^{(3)}(f) = 0$ otherwise; $Q^{(4)}(f) = 1$ if $f \in \langle x^2\rangle$, $Q^{(4)} = 1/4$ otherwise; $Q^{(5)}(f) = 1$ if $f \in \langle z\rangle$, $Q^{(5)}(f) = 1/4$ otherwise, $Q^{(6)}(f) = 1$ if $f \in \langle 0\rangle$, $Q^{(6)}(f) = 1/2$ otherwise. Then $Q^{(i)}$ is a fuzzy ideal of R, i = 1, ..., 6, such that $W = Q^{(1)} \cap Q^{(2)} \cap Q^{(3)}$ since $\langle x^2 + y^2 - 1, x^2z\rangle = \langle x^2, y - 1\rangle \cap \langle x^2, y + 1\rangle \cap \langle x^2 + y^2 - 1, z\rangle$, $X = Q^{(4)} \cap Q^{(5)}$ since $\langle x^2z\rangle = \langle x^2\rangle \cap \langle z\rangle$, $Y = Q^{(6)}$. Thus $A = \bigcap_{i=1}^{6} Q^{(i)}$ and in fact this is a irredundant fuzzy primary represntation of A. Now $\sqrt{Q^{(1)}}(f) = 1$ if $f \in \langle x, y - 1\rangle$, $\sqrt{Q^{(1)}}(f) = 0$ otherwise; $\sqrt{Q^{(2)}}(f) = 1$ if $f \in \langle x, y - 1\rangle$, $\sqrt{Q^{(2)}}(f) = 0$ otherwise; $\sqrt{Q^{(3)}} = Q^{(3)}$, $\sqrt{Q^{(4)}}(f) = 1$ if $f \in \langle x\rangle$, $\sqrt{Q^{(4)}}(f) = 0$ otherwise; $\sqrt{Q^{(5)}} = Q^{(5)}$, $\sqrt{Q^{(6)}} = Q^{(6)}$. Hence $\sqrt{A} = \bigcap_{i=1}^{6} P^{(i)}$ where $P^{(i)} = \sqrt{Q^{(i)}}$ is a fuzzy prime ideal of R, i = 1, ..., 6. We have the following fuzzy algebraic varieties:

$\mathcal{M}(P^{(1)})(b) = 1$ if $b \in \mathcal{M}(\langle x, y-1\rangle)$, $\mathcal{M}(P^{(1)})(b) = 0$ otherwise;

$\mathcal{M}(P^{(2)})(b) = 1$ if $b \in \mathcal{M}(\langle x, y + 1\rangle)$, $\mathcal{M}(P^{(2)}(b) = 0$ otherwise;

$\mathcal{M}(P^{(3)})(b) = 1$ if $b \in \mathcal{M}(\langle x^2 + y^2 - 1, z\rangle)$, $\mathcal{M}(P^{(3)})(b) = 0$ otherwise;

$\mathcal{M}(P^{(4)})(b) = c(1/4)$ if $b \in \mathcal{M}(\langle x\rangle)$, $\mathcal{M}(P^{(4)})(b) = 0$ otherwise;

$\mathcal{M}(P^{(5)})(b) = c(1/4)$ if $b \in \mathcal{M}(\langle z\rangle)$, $\mathcal{M}(P^{(5)})(b) = 0$ otherwise;

$\mathcal{M}(P^{(6)})(b) = c(1/2)\ \forall\, b \in L^k$.

Then $\mathcal{M}(A) = \bigcup_{i=1}^{6} \mathcal{M}(P^{(i)})$ and in fact $\mathcal{M}(P^{(i)})$ is irreducible and no $\mathcal{M}(P^{(i)})$ is contained in the union of the others, i = 1, ..., 6.

Consider the nonlinear system of equations of fuzzy singletons,

$$(x_s)^2 + (y_t)^2 - 1_{1/4} = 0_{1/4},$$
$$(x_s)^2 z_u = 0_{1/2}.$$

Then a solution is given by $t \geq 1/4$ and $\min\{s, u\} = 1/2$ and the solution of $x^2 + y^2 - 1 = 0$ and $x^2z = 0$. Note also that $A = \langle (x^2 + y^2 - 1)_{1/4}, (x^2z)_{1/2}\rangle$ by Theorem 2.8. If we let $c(0) = 1$, $c(1/4) = 1/2$, $c(1/2) = 1/4$, $c(1) = 0$, then the above representation of $\mathcal{M}(A)$ seems to better represent the solution of the above nonlinear system of equations of fuzzy singletons. The $\mathcal{M}(P^{(i)}$ for i = 1, 2, 3, yield the crisp part of the solution while the $\mathcal{M}(P^{(i)})$ for i = 4, 5, 6 yield the fuzzy part.

Proposition 2.10. Let S be a fuzzy subset of R. Then $<S>^* = <S^*>$.

Corollary 2.11. Let S be a fuzzy subset of R. Then $b \in L^k$ is a zero of S^* if and only if b is a zero of $<S>^*$.

Let W be a finite subset of S^* such that $<W> = <S^*>$ where S is a fuzzy subset of R. Define the fuzzy subset T of R by $T(b) = S(b)$ if $b \in W$ and $T(b) = 0$ otherwise. Then $T^* = W$. Hence $<T>^* = <T^*> = <S^*> = <S>^*$. Thus b is a zero of $<S>^*$ iff b is a zero of $<T>^*$.

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