SOME REMARKS ON FUZZY NORMAL AND COMPLETELY NORMAL SPACES

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Abstract : The interrelationship between various existing concepts of fuzzy normal and completely normal spaces are studied.

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1. Introduction : Separation axioms in fuzzy topological spaces (fts) have been discussed by several authors. In particular, two fuzzy versions of normality viz, normality and weak normality have been introduced by Kerre [3]. Fuzzy normal and completely normal spaces have been studied by Hu Cheng Ming [5]. Fora [2] has introduced two notions of fuzzy normal and four notions of fuzzy completely normal spaces. For any two fuzzy subsets μ, ν of a set X, $\mu \cap \nu = \emptyset \Longrightarrow \mu \leq 1-\nu$ but the reverse implication does not hold. This peculiarity of fuzzy sets leads to four possible definitions of fuzzy normal and fuzzy completely normal spaces viz. fn_i- and fcn_i-spaces for i = 1,2,3,4. It has been found that $fn_1 \Longleftrightarrow$ normal (Fora[2]), $fn_3 \Longleftrightarrow$ normal (Kerre[3]) $\Longleftrightarrow$ w-normal (Fora[2]), $fn_4 \Longleftrightarrow$ weakly normal (Kerre[3]), fcn_1 (or fcn_2 or fcn_3 or fcn_4) $\Longleftrightarrow$ ws (or ss or ww or sw)-completely normal (Fora[2]). Also $fcn_2 \Longleftrightarrow$ fuzzy completely normal (Hu Cheng

Ming [5]. The interrelationship between the different concepts of fuzzy normality and completely normality have been studied. The validity of the charaterization of completely normal spaces as hereditarily normal spaces has been examined in the fuzzy setting.

2. Preliminaries : Let $X = \{x_1, x_2, \ldots, x_n\}$ and $a_i \in I$ for $i = 1, 2., \ldots, n$. By $\{a_1, a_2, \ldots, a_n\}$, we mean the fuzzy subset λ of X s.t. $\lambda(x_i) = a_i$ for $i = 1, 2, \ldots, n$.

Unless otherwise mentioned, as regards fuzzy topological notions, we follow Ming & Ming [4].

The family of all fuzzy closed sets in a fuzzy topological space (X,τ) will be denoted by $C(\tau)$.

If two fuzzy subsets λ, δ of a set X be not quasi-coincident, then we write $\lambda \bar{q} \delta$.

Definition 2.1 : [5] Two fuzzy sets λ, δ of a set X are said to be strong quasi-discoincident (written as $\lambda \bar{q}_s \delta$) if $\lambda(x) + \delta(x) \leq 1$ and $\lambda(x) + \delta(x) = 1 \Longrightarrow \lambda(x) = 1$ or $\delta(x) = 1$.

Definition 2.2 : [3] A fts (X,τ) is said to be a normal (resp.weakly normal) space (henceforth called fn(K) (resp. fwn(K) space)) if $\forall$ $\lambda, \delta \in C(\tau)$ s.t. $\lambda \bar{q} \delta$ (resp. $\lambda \cap \delta = \emptyset$), $\exists$ $\mu, \nu \in \tau$ s.t. $\lambda \leq \mu$, $\delta \leq \nu$ and $\mu \bar{q} \nu$.

Definition 2.3: [5] A fts (X,τ) is said to be a fuzzy normal space (henceforth called fn(H)-space) iff $\forall$ $\lambda \in C(\tau)$ and $\forall$ open-nhd μ of λ , $\exists$ a $\delta \in \tau$ s.t. $\lambda \leq \delta^o \leq \bar{\delta} \leq \mu$ and δ^o is a nhd of λ μ is a nhd of $\bar{\delta}$.

Theorem 2.4 : [5] A fts (X,τ) is fuzzy normal iff $\forall$ $\lambda, \delta \in C(\tau)$ s.t. $\lambda \bar{q}_s \delta$, $\exists$ $\mu, \nu \in \tau$ s.t. $\lambda \leq \mu$, $\delta \leq \nu$ and $\mu \bar{q} \nu$.

Definition 2.5 : [5] A fts (X,τ) is said to be a fuzzy completely normal space (henceforth called fcn(H)- space) if $\forall\ \lambda \in I^X$ and $\forall$ nhd μ of λ s.t. $x_{\lambda(x)} \in \mu, \exists$ a $\nu \in \tau$ s.t. $\lambda \le \nu \le \bar{\nu} \le \mu$ and moreover ν is a nhd of λ and $x_{\bar{\nu}(x)} \in \mu\ \forall\ x \in X$.

Definition 2.6 :[5] λ, δ are said to be fuzzy separated iff $\bar{\lambda}\ \bar{q}_s \bar{\delta}$ and $\lambda\ \bar{q}_s \bar{\delta}$.

Theorem 2.7 :[5] A fts (X,τ) is fuzzy completely normal iff for any pair of fuzzy separated sets λ, δ, $\exists\ \mu, \nu \in \tau$ s.t. $\lambda \le \mu$, $\delta \le \nu$ and $\mu\ \bar{q}\ \nu$.

Remark 2.8 :[6] $\forall\ \lambda, \mu \in I^X$, $\bar{\lambda}\ \bar{q}\ \mu$, $\lambda\ \bar{q}\ \bar{\mu} \Longleftrightarrow \exists\ \nu, \delta \in \tau$ s.t. $\lambda \le \nu$, $\mu \le \delta$, $\lambda\ \bar{q}\ \delta$, $\mu\ \bar{q}\ \nu$.

Definition 2.9 :[2] A fts (X,τ) is said to be a normal (resp. w-normal)(henceforth called fn(F)(resp. fwn(F)))-space iff $\forall\ \lambda, \delta \in C(\tau)$ s.t. $\lambda\ \bar{q}\ \delta$, $\exists\ \mu, \nu \in \tau$ s.t. $\lambda \le \mu$, $\delta \le \nu$ and $\mu \cap \nu = \emptyset$ (resp. $\mu\ \bar{q}\ \nu$).

Definition 2.10 :[2] $\lambda, \delta \in I^X$ are said to be s(resp.w)-separated iff $\bar{\lambda} \cap \mu = \lambda \cap \bar{\delta} = \emptyset$ (resp.$\bar{\lambda}\ \bar{q}\ \delta$, $\lambda\ \bar{q}\ \bar{\delta}$).

Definition 2.11:[2] A fts (X,τ) is said to be

(i) a ss(resp.ws)-completely normal space if for any two s (resp.w)-separated sets λ, δ, $\exists\ \mu, \nu \in \tau$ s.t. $\lambda \le \mu$, $\delta \le \nu, \mu \cap \nu = \emptyset$

(ii) a sw(resp.ww)-completely normal space if for any two s (resp.w)-separated sets λ, δ, $\exists\ \mu, \nu \in \tau$ s.t. $\lambda \le \mu$, $\delta \le \nu, \mu\ \bar{q}\ \nu$.

3. Fuzzy normal spaces.

Definition 3.1 : A fts (X,τ) is said to be a fuzzy n_i-normal space (or simply a fn_i-space), i = 1,2,3,4 if $\forall\ \lambda, \delta \in C(\tau)$ respectively

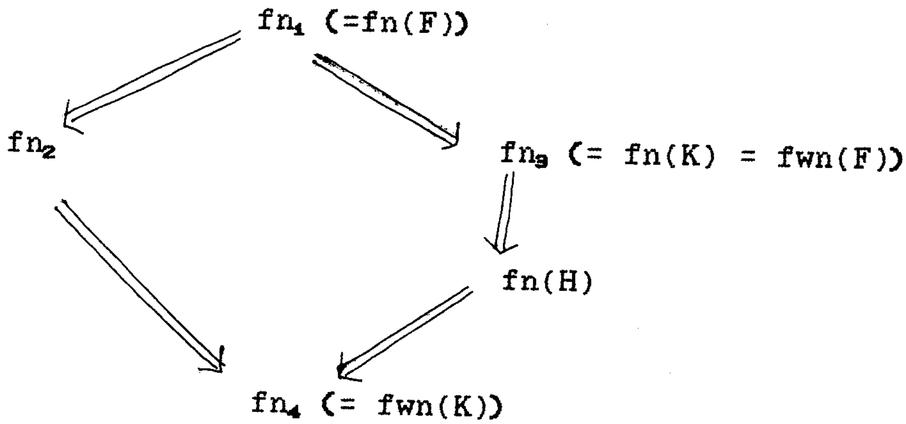
fn_1 : $\lambda\ \bar{q}\ \delta \Longrightarrow \exists\ \mu, \nu \in \tau$ s.t. $\lambda \le \mu$, $\delta \le \nu$, $\mu \cap \nu = \emptyset$.

fn_2 : $\lambda \cap \delta = \emptyset \Longrightarrow \exists\ \mu, \nu \in \tau$ s.t. $\lambda \le \mu$, $\delta \le \nu$, $\mu \cap \nu = \emptyset$.

fn_3 : $\lambda\ \bar{q}\ \delta \Longrightarrow \exists\ \mu, \nu \in \tau$ s.t. $\lambda \le \mu$, $\delta \le \nu$, $\mu\ \bar{q}\ \nu$.

fn_4 : $\lambda \cap \delta = \emptyset \Longrightarrow \exists\ \mu, \nu \in \tau$ s.t. $\lambda \le \mu$, $\delta \le \nu$, $\mu\ \bar{q}\ \nu$.

Remark 3.2 : Definitions 3.1,2.2,2.3 and 2.9 immediately yield the following implication diagram



To show that the reverse implications do not hold it follows from the above diagram that it is sufficient to show that $fn_2 \not\Longrightarrow fn(H)$ and $fn_3 \not\Longrightarrow fn_2$ which is evident from the following examples

Example 3.3: Let X = {a,b},

τ = {∅,1,(.7,∅),(∅,.7),(.7,1),(1,.7)(.7,.7),(.4,.4),(.7,.4),(.4,.7),(.4,∅),(∅,.4)}.

Then (X,τ) is a fn_2-space.

Let $\lambda = (.3,.3)$, $\delta = (.6,.6)$. Then $\lambda,\delta \in C(\tau)$ and $\lambda\ \bar{q}_s\ \delta$. But there do not exist two quasi-discoincident members of τ containing λ,δ. So (X,τ) is not a fn(H)-space.

Example 3.4 : Let X = {a,b}.

τ = {∅,1,(.4,.6),(.6,.4),(.4,.4),(.6,.6),(1,.6),(.6,1)}.

Then (X,τ) is a fn_3-space.

Let $\lambda = (.4,\emptyset)$, $\delta = (\emptyset,.4)$. Then $\lambda,\delta \in C(\tau)$ and $\lambda \cap \delta = \emptyset$. But there do not exist two disjoint members of τ containing λ,δ. So (X,τ) is not a fn_2 -space.

Theorem 3.5 [5] : A fts (X,τ) is a fn_3-space iff the following property is satisfied.

(P) : $\forall\ \lambda \in C(\tau)$ and $\forall\ \mu \in \tau$ s.t. $\lambda \le \mu$, $\exists$ a $\nu \in \tau$ s.t. $\lambda \le \nu$ and $\bar{\nu} \le \mu$.

Remark 3.6 : For a fts (X,τ) to be a

(i) fn_1-space, (P) is necessary but not sufficient;

(ii) fn_4(resp.fn(H))-space, (P) is sufficient but not necessary;

(iii) fn_2-space, (P) is neither necssary nor sufficient.

4. Fuzzy completely normal spaces

Definition 4.1: A fts (X,τ) is said to be a fuzzy completely n_i-normal space or simply a fcn_i-space for i = 1,2,3,4 if $\forall\ \lambda,\delta \in I^X$ respectively

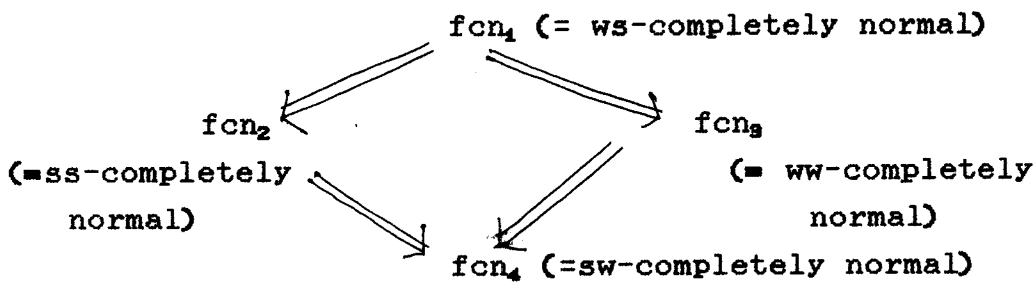
fcn_1 : $\bar{\lambda}\ \bar{q}\ \delta,\ \lambda\ \bar{q}\ \bar{\delta}$ ===> $\exists\ \mu,\nu \in \tau$ s.t. $\lambda \le \mu,\ \delta \le \nu,\ \mu \cap \nu = \emptyset$;

fcn_2 : $\bar{\lambda} \cap \delta = \emptyset = \lambda \cap \bar{\delta}$ ===> $\exists\ \mu,\nu \in \tau$ s.t. $\lambda \le \mu,\ \delta \le \nu,\ \mu \cap \nu = \emptyset$;

fcn_3 : $\bar{\lambda}\ \bar{q}\ \delta,\ \lambda\ \bar{q}\ \bar{\delta}$ ===> $\exists\ \mu,\nu \in \tau$ s.t. $\lambda \le \mu,\ \delta \le \nu,\ \mu\ \bar{q}\ \nu$;

fcn_4 : $\bar{\lambda} \cap \delta = \emptyset = \lambda \cap \bar{\delta}$ ===> $\exists\ \mu,\nu \in \tau$ s.t. $\lambda \le \mu,\ \delta \le \nu,\ \mu\ \bar{q}\ \nu$.

Remark 4.2 : Definitions 4.1 and 2.11 immediately yield the implication diagram



To show that the reverse implications do not hold it follows from the above diagram that it is sufficient to show that fcn_3 =#=> fcn_2 and fcn_2 =#=> fcn_3 which have been shown by the following examples.

Example 4.3 : Let X = {a,b,c},

τ = { Ø,1,(.4,Ø,.6),(Ø,.5,Ø),(.7,1,.4),(1,.5,1),(.4,.5,.6),
(.7,.5,.4),(.7,1,.6),(.4,.5,.4),(.7,.5,.6),(.4,Ø,.4)}.

Then (X,τ) is a fcn_2-space.

If λ = (.6,.5,.4), δ = (.3,.5,.6), then $\bar{\lambda}\ \bar{q}\ \delta$ and $\lambda\ \bar{q}\ \bar{\delta}$ .But there do not exist quasi-discoincident members of τ containing λ and δ. So (X,τ) is not a fcn_8-space.

Example 4.4 : Let $X = \{a,b\}$.

$\tau = \{\emptyset, 1, (.6,.4), (.4,.6), (.4,.4), (.6,.6), (1,.6), (.6,1)\}$.

Then (X,τ) is a fcn_8-space.

But since there do not exist any disjoint members of τ, (X,τ) is not a fcn_2-space.

Theorem 4.5 : A fts (X,τ) is a fcn_8-space iff it is a fcn(H)-space.

Proof: Definition 4.1 and Theorem 2.5 [5] yield that if (X,τ) be a fcn_8-space, then it is a fcn(H)-space.

Conversely, let (X,τ) be a fcn(H)-space.

Let $\lambda,\delta \in I^X$ be s.t. $\bar{\lambda}\ \bar{q}\ \delta$ and $\lambda\ \bar{q}\ \bar{\delta}$.

If $\bar{\lambda}\ \bar{q}_s\ \delta$ and $\lambda\ \bar{q}_s\ \bar{\delta}$,then the theorem is obvious.

Next let $\bar{\lambda}\ q_s\ \delta$ and $\lambda\ q_s\ \bar{\delta}$ do not hold.

So if $X_1 = \{x \in X;\ \bar{\lambda}(x)+\delta(x) = 1,\ \text{but}\ \bar{\lambda}(x) \neq \emptyset, \delta(x) \neq \emptyset\}$

$X_2 = \{x \in X;\ \lambda(x)+\bar{\delta}(x) = 1,\ \text{but}\ \lambda(x) \neq \emptyset,\ \bar{\delta}(x) \neq \emptyset\}$

then at least one of X_1, X_2 is non empty.

Let $\forall$ (+)ve integer $n > 2$, $\lambda_n, \delta_n : X \dashrightarrow I$ be defined by

$$\lambda_n(y) = \begin{cases} \lambda(y) & \text{if } y \notin X_2 \\ 1 - \delta(y) - (1 - \delta(y))/n & \text{if } y \in X_2 \end{cases}$$

$$\delta_n(y) = \begin{cases} \delta(y) & \text{if } y \notin X_1 \\ 1 - \lambda(y) - (1 - \lambda(y))/n & \text{if } y \in X_1 \end{cases}$$

Then $\bar{\lambda}_n\ \bar{q}_s\ \delta_n$ and $\lambda_n\ \bar{q}_s\ \bar{\delta}_n$.

Therefore, since (X,τ) is a fuzzy completely normal space, $\exists\ \mu_n, \nu_n \in \tau$ s.t. $\lambda_n \le \mu_n$, $\delta_n \le \nu_n$ and $\mu_n\ \bar{q}\ \nu_n$.

Let $\mu = \cup\{\mu_n;\ n > 2\}$ and $\nu = \cup\{\nu_n;\ n > 2\}$.

Then μ, $\nu \in \tau$, $\mu \bar{q} \nu$ and $\lambda \le \mu$, $\delta \le \nu$.

So (X,τ) is a fcn_3-space.

Lemma 4.6 : Let $Y \subset X$ and let $\lambda, \delta \in C(\tau_Y)$. Then

(i) $\lambda \bar{q} \delta \Longrightarrow \bar{\lambda} \bar{q} \delta$, $\lambda \bar{q} \bar{\delta}$ in (X,τ);

(ii) $\lambda \cap \delta = \emptyset \Longrightarrow \bar{\lambda} \cap \delta = \lambda \cap \bar{\delta} = \emptyset$ in (X,τ)

where λ, δ denote the fuzzy closures of λ, δ respectively in (X,τ).

Proof : Since $\lambda, \delta \in C(\tau_Y)$, $\exists\ \lambda_1, \delta_1 \in C(\tau)$ s.t. $\lambda = \lambda_1 \cap Y$, $\delta = \delta_1 \cap Y$.

(i) Let $\lambda \bar{q} \delta$. Then $\lambda_1 \bar{q} \delta$. For if $x \in Y$, then $\lambda_1(x)+\delta(x) = \lambda(x)+\delta(x) \le 1$ (since $\lambda \bar{q} \delta$) and if $x \in X - Y$, then $\lambda(x)+\delta(x)$ $\emptyset+\delta(x) \le 1$. Similarly $\lambda \bar{q} \delta_1$. Thus $1 - \lambda_1$, $1 - \delta_1 \in \tau$, $\lambda \le 1 - \delta_1$, $\delta \le 1 - \lambda_1$, $\lambda \bar{q} (1 - \lambda_1)$, $\delta \bar{q} (1 - \delta_1)$.

So by Remark 2.7, $\bar{\lambda} \bar{q} \delta$ and $\lambda \bar{q} \bar{\delta}$ in (X,τ).

(ii) Let $\lambda \cap \delta = \emptyset$. Then $\forall\ x \in Y$, $(\bar{\lambda} \cap \delta)(x) = \min.\{\bar{\lambda}(x), \delta(x)\} = \min.\{\bar{\lambda}_Y(x), \mu(x)\} = \min.\{\lambda(x), \mu(x)\}$ (since λ is fuzzy closed in (Y,τ_Y)) $= \emptyset$ and $\forall\ x \in X - Y$, $(\bar{\lambda} \cap \delta)(x) = \emptyset$, since $\mu(x) = \emptyset$. Thus $(\bar{\lambda} \cap \delta) = \emptyset$. Similarly $\lambda \cap \bar{\delta} = \emptyset$.

Lemma 4.7 : Let $Y \subset X$ and $\lambda, \delta \in C(\tau_Y)$. If $\mu, \nu \in \tau$ s.t. $\lambda \le \mu$, $\delta \le \nu$, $\mu \cap \nu = \emptyset$(resp. $\mu \bar{q} \nu$), then $\mu_1 = \mu \cap Y$, $\nu_1 = \nu \cap Y \in \tau_Y$ and $\lambda \le \mu_1$, $\delta \le \nu_1$, $\mu_1 \cap \nu_1 = \emptyset$ (resp. $\mu_1 \bar{q} \nu_1$).

Theorem 4.8 : Every subspace of a fcn_i-space is a fn_i-space for $i = 1,2,3,4$.

The result follows from lemma 4.6 (i),(ii) and lemma 4.7.

Remark 4.9 : Fora [2] has proved that a hereditarily fn_3 - space is a fcn_4-space. Since $fn_1 \Longrightarrow fn_3$, a hereditarily fn_1-space is also a fcn_4-space. But a hereditarily fn_i-space is not a fcn_j-space as shown by the following examples

Example 4.10 : Let $X = \{a,b,c\}$,

$\tau = \{\emptyset, 1, (.3, .4, .2), (.6, .5, .2)\}$.

$C(\tau) = \{\emptyset, 1, (.7, .6, .8), (.4, .5, .8)\}$

(X,τ) is a hereditarily fn_i-space for $i = 1,2,3,4$.

If $\lambda = (.3, .5, .2)$, $\delta = (.4, .5, .2)$, then $\bar{\lambda}\ \bar{q}\ \delta$ and $\lambda\ \bar{q}\ \bar{\delta}$. But there do not exist two quasi-discoincident members of τ containing λ and δ.

So (X,τ) is not a fcn_3-space and therefore it is not a fcn_4-space.

Example 4.11 ; Let $X = \{a,b,c,d\}$,

$$\tau = \{\emptyset, 1, (.6,.6,1,.6), (1,.6,.6,.6), (1,.6,1,.6), (.6,.6,.6,.6), (.6,\emptyset,\emptyset,.6), (\emptyset,\emptyset,.6,.6), (.6,\emptyset,.6,.6), (\emptyset,\emptyset,\emptyset,.6)\}.$$

(X,τ) is a hereditarily fn_2-space.

If $\lambda = (.2,\emptyset,\emptyset,\emptyset)$ and $\delta = (\emptyset,\emptyset,.2,\emptyset)$, then $\bar{\lambda} \cap \delta = \emptyset = \lambda \cap \bar{\delta}$. But there do not exist two quasi-discoincident members of τ containing λ and δ.

So (X,τ) is not a fcn_4-space and therefore it is not a fcn_i-space for $i = 1,2,3$.

Example 4.12 : Let $X = \{a,b,c,d\}$,

$$\tau = \{\emptyset, 1, (\emptyset,\emptyset,.4,.4), (\emptyset,.4,.4,.4), (\emptyset,.6,.4,.4), (\emptyset,1,.4,.4) (.4,\emptyset,.4,.4), (.4,.4,.4,.4), (.4,.6,.4,.4), (.4,1,.4,.4), (.6,\emptyset,.4,.4), (.6,.4,.4,.4), (.6,.6,.4,.4), (.6,1,.4,.4), (1,\emptyset,.4,.4), (1,.4,.4,.4), (1,.6,.4,.4), (1,1,.4,.4)\}.$$

Then (X,τ) is a hereditarily fcn_i-space for $i = 1,2,3,4$.

If $\lambda = (.4,\emptyset,\emptyset,\emptyset)$, $\delta = (\emptyset,.4,\emptyset,\emptyset)$, then $\bar{\lambda} \cap \delta = \lambda \cap \bar{\delta} = \emptyset$. But there do not exist two disjoint members of τ containing λ, δ.

So (X,τ) is not a fcn_2-space.

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