Fuzzy Topology and Frames.

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This note summarises results obtained in [5] and [9]. Using ideas from the theory of frames (or locales) we construct a functor from a category of L-fuzzy topological spaces to the category of topological spaces. The special case where L is a continuous lattice is then investigated and we suggest good definitions of compactness and of the Hausdorff property for L-fuzzy spaces. These can be related to properties of the associated classical topological space. Finally we show that when L is completely distributive then any compact Hausdorff L-fuzzy space is topological. We begin by reviewing some definitions and notation from [3].

A *frame* is defined to be a complete Heyting algebra, or equivalently a complete lattice L satisfying the infinite distributive law: for any $a \in L$ and any $S \subseteq L$

$$a \wedge \left(\bigvee S\right) = \bigvee \{a \wedge x | x \in S\}.$$

In particular the following are examples of frames:

1. the interval $I = [0,1]$ of $\boldsymbol{R}$;
2. any finite distributive lattice;
3. any complete Boolean algebra;
4. any complete totally ordered set;
5. the open sets of a topological space ordered by set inclusion (here the intersection of a family $(U_i)_{i \in I}$ of open sets is the union of all open sets which contain $\bigcap_{i \in I} U_i$).

A *frame morphism* is defined to be a map between frames which preserves finite meets and arbitrary joins. A *point* of a frame L is a frame morphism from L to the two element frame $\{0, 1\}$. It can be shown that there is a 1-1 correspondence between the points of L and the prime elements of L, where $p \in L$ is called *prime* if $p \neq 1$ and whenever $x \wedge y \leq p$ then $x \leq p$ or $y \leq p$. The set of all prime elements of L will be denoted by $\mathrm{pt}(L)$.

A frame L is *spatial* if for all $a, b \in L$ with $a \not\leq b$ there is a $p \in \mathrm{pt}(L)$ with $a \not\leq p \geq b$. For example any complete totally ordered set is a frame in which every element except the largest is prime. It follows that a complete totally ordered set is a spatial frame. In particular the interval $I = [0,1]$ of $\boldsymbol{R}$ is a spatial frame for which $\mathrm{pt}(I) = [0,1)$.

If L is a lattice and X is a set then a map from X to L is called an *L-fuzzy subset* of X. We shall denote the set of all L-fuzzy subsets of X by $\mathcal{F}(X, L)$.

When L is a frame then $\mathcal{F}(X, L)$ is also a frame (when ordered pointwise). For each $a \in X$ and each $p \in \mathrm{pt}(L)$ let $a_p : X \to L$ be defined by

$$a_p(x) = \begin{cases} p & \text{if } x = a \\ 1 & \text{otherwise.} \end{cases}$$

Warner [8] has shown that these maps a_p are the points of the frame $\mathcal{F}(X, L)$ and calls them the L-fuzzy points of X. So the points of $\mathcal{F}(X, L)$ may be identified with the elements (a, p) of $X \times \mathrm{pt}(L)$.

When L is a frame then the map $\varphi : L \to \boldsymbol{P}(\mathrm{pt}(L))$ defined by

$$\varphi(x) = \{p \in \mathrm{pt}(L) | p \not\geq x\} \qquad \forall x \in L$$

is known to be a frame morphism which is injective if and only if L is spatial. Its image is therefore a topology on $\mathrm{pt}(L)$ [3, p.42]. Hence any spatial frame is frame isomorphic to a topology. Combining this with the above remarks we can prove the following [5].

Theorem. *Let L be a frame and $\mathcal{T}$ be an L-fuzzy topology on a set X, then the map $\phi : \mathcal{T} \to \boldsymbol{P}(X \times \mathrm{pt}(L))$ defined by*

$$\phi(\delta) = \{(a, p) | p \not\geq \delta(a)\} \qquad \forall \delta \in \mathcal{T}$$

is a frame morphism and $\phi(\mathcal{T})$ is a topology on $X \times \mathrm{pt}(L)$. If L is spatial then $\mathcal{T}$ is frame isomorphic to the topology $\phi(\mathcal{T})$.

Versions of this theorem have been proved by Lowen [4] for the case $L = I$ and by Eroğlu, [1] for a completely distributive totally ordered lattice L.

For an L-fuzzy point a_p and an L-fuzzy subset f of X we write $a_p \boldsymbol{\in} f$ if $f(a) \not\leq p$. This gives the following relationship between fuzzy membership $\boldsymbol{\in}$ and ordinary membership $\in$:

$$a_p \boldsymbol{\in} \delta \iff p \not\geq \delta(a) \iff (a, p) \in \phi(\delta).$$

Let L be a frame. If $(X, \mathcal{T})$ is an L-fuzzy topological space we shall denote the topological space $(X \times \mathrm{pt}(L), \phi(\mathcal{T}))$ obtained in the above theorem by

$\Phi(X, \mathcal{T})$. For each map $f : X \to Y$ define a map $\Phi(f) : X \times \mathrm{pt}(L) \to Y \times \mathrm{pt}(L)$ by

$$\Phi(f)(x,p) = (f(x),p) \qquad \forall x \in X, \forall p \in \mathrm{pt}(L).$$

If $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ are L-fuzzy topological spaces. then a map $f : X \to Y$ is called *continuous* if $\delta \circ f \in \mathcal{S}$ for every $\delta \in \mathcal{T}$. We shall denote by L-**Top** the category whose objects are L-fuzzy topological spaces and whose arrows are the corresponding continuous maps. The category of topological spaces with continuous maps will be denoted by **Top**.

Proposition. *The map Φ is a functor from L-**Top** to **Top**. If $\mathrm{pt}(L)$ is nonempty then Φ is faithful. If L is spatial then Φ is full.*

In certain cases the topology $\phi(\mathcal{T})$ is the product of a topology on X with the topology $\varphi(L)$. Before stating our theorem concerning this we review some definitions from [2].

A subset S of a complete lattice L is called *Scott open* if it satisfies
(1) if $a \in S$ then $x \in S$ for all $x \geq a$;
(2) if D is a directed set with $\bigvee D \in S$ then $d \in S$ for some $d \in D$.

The set of all Scott open subsets of L is a topology, called the *Scott topology* of L. If $a, b \in L$ we write $a \ll b$ if b lies in the interior of the set $\{x \in L | x \geq a\}$. A *continuous lattice* is defined to be a complete lattice L in which $a = \bigvee \{x \in L | x \ll a\}$ for every $a \in L$. Warner ([7]) proves the following result.

Proposition. *For a continuous frame L the set of continuous functions from a topological space (X, τ) to L with the Scott topology, forms an L-fuzzy topology (which will be denoted by $\omega(\tau)$).*

For example the interval I of $\mathbb{R}$ is a continuous frame. In this case if (X, τ) is a topological space then $\omega(\tau)$ is the set of all lower semicontinuous functions $X \to I$.

Theorem. *Let L be a continuous frame, let (X, τ) be a topological space and let $\omega(\tau)$ be the fuzzy topology of continuous maps from X to L where L has its Scott topology. Then the topology $\phi(\omega(\tau))$ on $X \times \mathrm{pt}(L)$ is the product topology $\tau \times \varphi(L)$.*

This theorem provides a "goodness of extension" criterion for L-fuzzy topological properties.

We call an L-fuzzy topological space $(X, \mathcal{T})$ *Hausdorff* if for every $p, q \in \mathrm{pt}(L)$ and every pair x, y of distinct elements of X, there exist $f, g \in \mathcal{T}$ with

$$x_p \in f, \quad y_q \in g, \quad \text{and} \quad (\forall z \in X)\ f(z) = 0 \text{ or } g(z) = 0.$$

This definition is good i.e. if L is a continuous lattice and (X, τ) is a topological space then (X, τ) is Hausdorff if and only if the L-fuzzy space $(X, \omega(\tau))$ is Hausdorff. Our disjointness condition is justified by the fact that for any spatial local L the following are equivalent:
(1) $(\forall z \in X)\ f(z) = 0$ or $g(z) = 0$;
(2) the subsets $\phi(f)$ and $\phi(g)$ of $X \times \mathrm{pt}(L)$ are disjoint.

An L-fuzzy topological space $(X, \mathcal{T})$ is said to be *compact* if for every prime p of L and every collection $(f_i)_{i \in I}$ of open L-fuzzy sets with $\left(\bigvee_{i \in I} f_i\right)(x) \not\leq p$ for all $x \in X$, there is a finite subset F of I with $\left(\bigvee_{i \in F} f_i\right)(x) \not\leq p$ for all $x \in X$.

When L is a continuous lattice we can prove the following relations between fuzzy and classical compactness. If $(X, \mathcal{T})$ is an L-fuzzy space then these are equivalent:

(1) $(X, \mathcal{T})$ is fuzzy compact;
(2) for every $p \in \mathrm{pt}(L)$, $X \times \{p\}$ is a compact subspace of $(X \times \mathrm{pt}(L), \phi(\mathcal{T}))$;
(3) for every $p \in \mathrm{pt}(L)$, $X \times \{q \in \mathrm{pt}(L) | q \leq p\}$ is a compact subspace of $(X \times \mathrm{pt}(L), \phi(\mathcal{T}))$.

The above definition of L-fuzzy compactness is good, i.e. if L is a continuous lattice and (X, τ) is a topological space then (X, τ) is compact if and only if the L-fuzzy space $(X, \omega(\tau))$ is compact.

Finally we can show that when L is a completely distributive lattice then every compact Hausdorff L-fuzzy topology containing all constant functions is topological. Our proof is based on [6] who deal with the special case $L = I$.

Theorem. *Let L be a completely distributive lattice and let $(X, \mathcal{T})$ be a compact Hausdorff L-fuzzy space where $\mathcal{T}$ contains all constant functions from X to L. Then $(X, \mathcal{T})$ is topological i.e. there is a topology τ on X with $\mathcal{T} = \omega(\tau)$.*

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