FURTHER STUDY ON THE CONVERGENCE THEOREMS FOR CONORMED-SEMINORMED FUZZY INTEGRAL

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Abstract In this paper, the X(p) condition to the pair of t-conorm and t-seminorm is introduced and two monotone convergence theorems and a uniform convergence theorem for comormed-seminormed fuzzy integral are obtained in X(p) conditions.

1. Introduction

Suarez and Gil [1] firstly proposed the concept of conormed-seminormed fuzzy integral which is the extensions of Sugeno fuzzy integral and classcal Lebesgue integral(when the fuzzy measure is probability measure). Liu[2] proved that when the t-conorm and t-seminorm are continuous, the monotone convergence theorem to the case that $f_n \uparrow f$ is true, and used an example to explain that the monotone convergence theorem to the case that $f_n \downarrow f$ is not true in general, even in the condition that the t-conorm and t-seminorm are continuous. In this paper, motivatied by Yang[3], we introduce the X(p) condition to the pair of (T', S), where T' is a t-conorm and S a t-seminorm, and give many examples of (T', S) which satisfy X(p) conditions, then, in the assumption that (T', S) satisfies X(p) condition, we obtain two monotone convergence theorems to monotone convergence measurable function sequence $\{f_n\}$, $f_n \uparrow f$(resp.

$f_n \downarrow f$) in the condition that f is the (c)control function of $\{f_n\}$, and a uniform convergence theorem to uniformly convergent measurable function sequence $\{f_n\}$, $f_n \to f$ in the condition that $\inf_{x \in X} f(x) > 0$.

The concepts and notations not defned in this paper can be found in [1].

2. X(p) conditions

Definition2.1 Let T' be a t-conorm and S a t-seminorm. If there exists $p \geq 1$ such that $\forall\, c \in (0, 1)$, $\forall\, n \in N := \{1, 2, \ldots\}$, $\forall\, a_i$, $b_i \in [0, 1]$ $(i = 1, 2, \ldots, n)$, there holds

$$(2.1) \qquad \mathop{T'}_{i=1}^{n} [S(ca_i, b_i)] \geq c^p \mathop{T'}_{i=1}^{n} [S(a_i, b_i)]$$

we call (T', S) satisfies X(p) condition.

About X(p) condition, we have the following two propositions.

Proposition2.1 (T', S) satisfies X(1) condition iff (2.1) holds for n = 2.

Proof It is enough to prove the sufficiency.

(1) The case that n = 1

$\forall\, c \in (0, 1)$, $\forall\, a_1, b_1 \in [0, 1]$, we have

$$\mathop{T'}_{i=1}^{1} [S(ca_i, b_i)] = S(ca_1, b_1) = T'[S(ca_1, b_1), 0]$$
$$= T'[S(ca_1, b_1), S(0, 0)] = T'[S(ca_1, b_1), S(c0, 0)]$$
$$\geq cT'[S(a_1, b_1), S(0, 0)] = cT'[S(a_1, b_1), 0] = cS(a_1, b_1)$$
$$= c \mathop{T'}_{i=1}^{1} [S(a_i, b_i)]$$

(2) The case that $n \geq 2$

Now we use the mathematical induction to prove the conclusion.

The conclusion for n = 2 is the assumption.

Suppose that the conclusion is true for $n (\geq 2)$, we prove that the conclusion is true for n+1.

$\forall c \in (0, 1)$, $\forall a_i, b_i \in [0, 1]$, $i = 1, 2, \ldots, n+1$, we have

$$\begin{aligned}
\overset{n+1}{\underset{i=1}{T'}} [S(ca_i, b_i)] &= T'\{\overset{n}{\underset{i=1}{T'}} [S(ca_i, b_i)], S(ca_{n+1}, b_{n+1})\} \\
&\geq T'\{c\overset{n}{\underset{i=1}{T'}} [S(a_i, b_i)], S(ca_{n+1}, b_{n+1})\} \\
&\geq T'\{c\overset{n}{\underset{i=1}{T'}} [S(a_i, b_i)], cS(a_{n+1}, b_{n+1})\} \qquad \text{(from (1))} \\
&= T'\{S(c\overset{n}{\underset{i=1}{T'}} [S(a_i, b_i)], 1), S(cS(a_{n+1}, b_{n+1}), 1)\} \\
&\geq cT'\{S(\overset{n}{\underset{i=1}{T'}} [S(a_i, b_i)], 1), S(S(a_{n+1}, b_{n+1}), 1)\} \\
&= cT'\{\overset{n}{\underset{i=1}{T'}} [S(a_i, b_i)], S(a_{n+1}, b_{n+1})\} = c\overset{n+1}{\underset{i=1}{T'}} [S(a_i, b_i)] \qquad \square
\end{aligned}$$

Proposition2.2 (V, S) (where V:= max) satisfies X(p) condition ($p \geq 1$) iff (2.1) holds for $n = 2$.

Proof Like Proposition2.1, it is enough to prove the suffciency.

(i) The proof to the case that $n = 1$ is similar to the (1) of the proof of Proposition2.1.

(ii) The case that $n \geq 2$

We use mathematical induction to prove the conclusion.

The conlusion for $n = 2$ is the assumption.

Suppose that the conclusion is true for $n(\geq 2)$, we prove the conclusion is true for $n+1$.

$\forall c \in (0, 1)$, $\forall a_i, b_i \in [0, 1]$, $i = 1, 2, \ldots, n+1$, we have

$$\begin{aligned}
\overset{n+1}{\underset{i=1}{V}} [S(ca_i, b_i)] &= V\{\overset{n}{\underset{i=1}{V}} [S(ca_i, b_i), S(ca_{n+1}, b_{n+1})\} \\
&\geq V\{c^p \overset{n}{\underset{i=1}{V}} S(a_i, b_i), S(ca_{n+1}, b_{n+1})\} \\
&\geq V\{c^p \overset{n}{\underset{i=1}{V}} S(a_i, b_i), c^p S(a_{n+1}, b_{n+1})\} \qquad \text{(from (i))} \\
&= c^p\, V\{\overset{n}{\underset{i=1}{V}} S(a_i, b_i), S(a_{n+1}, b_{n+1})\} \\
&= c^p \overset{n+1}{\underset{i=1}{V}} S(a_i, b_i) \qquad \square
\end{aligned}$$

In the following, we give some example of (T', S) which satisfy X(p) conditions.

Example2.1 Note

$S_1(x, y) := x \wedge y$,

$S_2(x, y) := xy$,

$$T'_1(x, y) := x \vee y,$$

$$T'_{2,p}(x, y) := (x^p + y^p)^{\frac{1}{p}} \wedge 1 \qquad (p \geq 1)$$

By using Proposition2.1, we know that (T'_1, S_1), (T'_1, S_2), $(T'_{2,p}, S_1)$ and $(T'_{2,p}, S_2)$ satisfies X(1) conditions.

Note. (T'_1, S_1) is to Suguno's fuzzy integral and $(T'_{2,p}, S_2)$ to classical Lebesgue integral(when the fuzzy measure is a probability measure).

Example2.2 For $p \geq 1$, note

$$S^1_p(x, y) = x^p \wedge y, \qquad x < 1,\ y < 1,$$
$$= x \wedge y, \qquad \text{else}$$
$$S^2_p(x, y) = x^p y \qquad x < 1,\ y < 1$$
$$= xy \qquad \text{else}$$

By using Proposition2.2, we know that (V, S^1_p) and (V, S^2_p) satisfies X(p) conditions.

Note. The t-seminorms S^1_p and S^2_p are not continuous when $p > 1$.

3. Monotone convergence theorems for conormed-seminormed fuzzy integral in X(p) conditions.

In this section, we give two monotone convergence theorems for conormed-seminormed fuzzy integral in X(p) conditions when the monotone measurable function sequence $\{f_n\}$ has (c)control functions.

Definition3.1 Let $\{f_n\}$ be a monotone measurable function sequence, $f_n \uparrow f$(resp. $f_n \downarrow f$). We call f the (c)control function of $\{f_n\}$, if $\forall\ c \in (0, 1)$, there exists $n(c) \in N$ such that

$$f_n(x) \geq cf(x) \qquad \forall\ x \in X$$

(resp. $cf_n(x) \leq f(x) \qquad \forall\ x \in X$)

whenever $n \geq n(c)$.

Theorem3.1 If (T', S) satisfies $X(p)$ condition and $f_n \uparrow f$, f is the (c)control function of $\{f_n\}$, then

$$\lim_{n\to\infty} \int_{X}^{T'S} f_n(x).g(.) = \int_{X}^{T'S} f(x).g(.)$$

Proof Note

$$\int_{X}^{T'S} f(x).g(.) := k$$

It is clear that

$$\lim_{n\to\infty} \int_{X}^{T'S} f_n(x).g(.) \leq k$$

Now, we prove that

$$\lim_{n\to\infty} \int_{X}^{T'S} f_n(x).g(.) \geq k$$

Since f is the (c)control functionof $\{f_n\}$, then, $\forall\, c \in (0, 1)$, there exists $n(c) \in N$ such that

$$f_n(x) \geq cf(x) \qquad \forall\, x \in X$$

whenever $n \geq n(c)$.

$\forall$ fixed $n \geq n(c)$, $\forall$ simple function $s = \sum_{i=1}^{m} \alpha_i \mu_{A_i}$, $0 \leq s \leq f$, we have

$$\int_{X}^{T'S} f_n(x).g(.) \geq \int_{X}^{T'S} (cf)(x).g(.) \geq Q_{T'S}^{X}(cs)$$

$$= T'^{m}_{i=1}[S(c\alpha_i,\ g(A_i))]$$

$$\geq c^{p}\ T'^{m}_{i=1}\ [S(\alpha_i,\ g(A_i))] \quad \text{(from X(p) condition)}$$

$$= c^{p} Q_{T'S}^{X}(s)$$

Therefore,

$$\int_{X}^{T'S} f_n(x).g(.) \geq c^{p} \int_{X}^{T'S} f(x).g(.) = c^{p} k$$

Consequently,

$$\lim_{n\to\infty} \int_{X}^{T'S} f_n(x).g(.) \geq c^{p} k$$

Since $c \in (0, 1)$ is arbitary, then

$$\lim_{n\to\infty} \int_{X}^{T'S} f_n(x).g(.) \geq k \qquad \square$$

Similarly, we can prove

Theorem3.2 If (T', S) satisfies $X(p)$ condition and $f_n \downarrow f$, f is the (c)control function of $\{f_n\}$, then

$$\lim_{n\to\infty} \int_{X\,T'S} f_n(x).g(.) = \int_{X\,T'S} f(x).g(.)$$

Note. In Theorem3.2, the condition that f is the (c)control function of $\{f_n\}$ can not be dropped. we can use the following example to explain it(cf. [2]).

Example3.1 Let $(T', S) := (T_{2,1}, S_1)$(see section2), $(X, \mathcal{F}, g)$: $= ([0, 1], [0, 1]\cap\mathcal{B}^1, m)$(where m is Lebesgue measure on Borel field $[0, 1]\cap\mathcal{B}^1$, $f_n := 1/n$, $\forall\ n\in N$, $f := 0$, we have, $\forall\ n\in N$,

$$\int_{X\,T'S} f_n(x).g(.) = 1$$

but

$$\int_{X\,T'S} f(x).g(.) = 0$$

amd then

$$\lim_{n\to\infty} \int_{X\,T'S} f_n(x).g(.) \neq \int_{X\,T'S} f(x).g(.)$$

In this paper, the following problem can not be solved: can the condition that f is the (c)control function of $\{f_n\}$ be dropped in Theorem3.1.

4. Uniform convergence theorem for conormed-seminormed fuzzy integral in X(p) conditions

The main result of this section is

Theorem4.1 If (T', S) satisfies X(p) condition and the measureable function sequence $\{f_n\}$ converges uniformly to f, $\inf_{x\in X} f(x) := I > 0$, then

$$\lim_{x\in X} \int_{X\,T'S} f_n(x).g(.) = \int_{X\,T'S} f(x).g(.)$$

Lemma4.1 In the assumptions of Theorem4.1, we have

$$f_n \uparrow f \Rightarrow \lim_{n\to\infty} \int_{X\,T'S} f_n(x).g(.) = \int_{X\,T'S} f(x).g(.)$$

Proof Since $f_n \uparrow f$ and f converges to f uniformly, then, $\forall\ c\in(0, 1)$, there exists $n(c)\in N$ such that

(4.1) $\quad f(x) - f_n(x) \leq (1 - c)I \qquad \forall\ x\in X$

as $n \geq n(c)$.

From (4.1), we know that

(4.2) $\quad f_n(x) \geq f(x) - (1 - c)I \geq I - (1 - c)I = cI \qquad \forall\ x \in X$

as $n \geq n(c)$.

Therefore, $\forall\ c \in (0, 1)$, from (4.1) and (4.2), we have

$$f_n(x) - cf(x) = (1-c)f_n(x) - c(f(x) - f_n(x))$$
$$\geq (1-c)cI - c(1-c)I = 0$$

i. e.

(4.3) $\quad f_n(x) \geq cf(x) \qquad \forall\ x \in X$

as $n \geq n(c)$.

(4.3) yields that f is the (c)control function of $\{f_n\}$ and the conclusion follows from Theorem3.1. □

Lemma4.2 In the assumptiona of Theorem4.1, we have

$$f_n \downarrow f \Rightarrow \lim_{n\to\infty} \int_X^{T's} f_n(x).g(.) = \int_X^{T's} f(x).g(.)$$

Proof Since $f_n \downarrow f$ and f_n converges to f uniformly, then, $\forall\ c \in (0, 1)$, there exists $n(c) \in N$ such that

(4.4) $\quad f_n(x) - f(x) \leq [(1-c)/c]I \qquad \forall\ x \in X$

as $n \geq n(c)$.

Therefore,

$$f(x) - cf_n(x) = (1-c)f(x) - c(f_n(x) - f(x))$$
$$\geq (1-c)I - c[(1-c)/c]I = 0 \qquad \forall\ x \in X$$

i. e.

(4.5) $\quad f(x) \geq cf\ (x) \qquad \forall\ x \in X$

as $n \geq n(c)$.

(4.5) tells us that f is the (c)control function of $\{f_n\}$ and the conclusion follows from Theorem3.2. □

Proof of Theorem4.1

$\forall\ n \in N$, note

$$\bar{h}_n(x) := \bigvee_{k \geq n} f_k(x) \qquad \forall\ x \in X$$

$$\underline{h}_n(x) := \bigwedge_{k \geq n} f_k(x) \qquad \forall\ x \in X$$

we can easily obtain that $\forall\ n \in N$, $\underline{h}_n \leq f \leq \bar{h}_n$, $\underline{h}_n \uparrow f$, $\bar{h}_n \downarrow f$, and $\{\underline{h}_n\}$, $\{\bar{h}_n\}$ converge to f uniformly.

From Lemma4.1 and 4.2, we obtain

$$(4.6) \qquad \lim_{n\to\infty} \int_{X}^{T's} \underline{h}_n(x).g(.) = \int_{X}^{T's} f(x).g(.)$$

and

$$(4.7) \qquad \lim_{n\to\infty} \int_{X}^{T's} \bar{h}_n(x).g(.) = \int_{X}^{T's} f(x).g(.)$$

From $\underline{h}_n \leq f_n \leq \bar{h}_n$, $\forall\ n \in N$, we have, $\forall\ n \in N$,

$$(4.8) \qquad \int_{X}^{T's} \underline{h}_n(x).g(.) \leq \int_{X}^{T's} f_n(x).g(.) \leq \int_{X}^{T's} \bar{h}_n(x).g(.)$$

(4.6), (4.7) and (4.8) yield that

$$\lim_{n\to\infty} \int_{X}^{T's} f_n(x).g(.) = \int_{X}^{T's} f(x).g(.) \qquad \square$$

Note. The condition trhat $\inf_{x\in X} f(x) > 0$ can not be dropped in Theorem4.1. we can use Example3.1 to explain it.

References

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