Entropy of Fuzzy Measure Spaces

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Abstract

In this paper, we first introduce the concept of associated probabilities to a fuzzy measure, This development is based on the consideration of shannon entropy of fuzzy measure spaces. Then, some basic properties of entropy of fuzzy measure spaces are discussed.

Keywords: fuzzy measure, associated probability, entropy, information space.

§ 1. Introduction

In information theory, the entropy of an information source is an important concept, when the information is expressed by applying fuzzy measure, additivi-ty does not seem suitable as a demandable property of set functions. In many real situations, it is sometimes more appropriate to consider non—additive but monotone valuations to express some information. So the study of entropy of fuzzy measure spaces is very useful. But, if fuzzy measure spaces are different, the degree of uncertainty of fuzzy measure spaces are also differ-ent. In order to measure the degree of uncertainty of a fuzzy measure space by a real number, in this paper, We introduce the concept of entropy of fuzzy measure spaces.

The relationships between fuzzy measure and additive measure have been studied in [2、3]. In this paper, We first introduce the concept of associated probabilities to a fuzzy measure, This development is based on the consider -ation of Shannon entropy of fuzzy measure spaces. Then, some basic properties of entropy of fuzzy measure spaces are discussed. In last section, we also study the entropy of a information space.

§2. Associated Probabilities to a Fuzzy Measure

Let $X=\{x_1\cdots\cdots x_n\}$, be a finite set and m a set function

$$m: P(X) \rightarrow [0,1]$$

Where P(X) is the power set of X. We will say m is a fuzzy measure on X if it satisfies:

(i) $m(\emptyset)=0$; $m(X)=1$.

(ii) $\forall A, B \subseteq X$, if $A \subseteq B$ then $m(A) < m(B)$

the triple (X, P(X), m) is called a fuzzy measare space,

Definition2. 1: Let m be a fuzzy measure on (X, P(X)), The fuzzy measure m^* defined by

$$m^*(A) = 1 - m(\bar{A}) \qquad \forall A \subseteq X \tag{2.1}$$

is called the dual fuzzy measure of m,

Where $\bar{A}$ is the complement of A.

Particularly, a fuzzy measare is said to be autodual if it conincides with its dual measare, that is, $m^*(A) = m(A)$ $\forall A \subseteq X$.

The possible order of the elements of X are given by the permutations of a set with n elemnts. Which form the group Sn. For any $\sigma = (\sigma(1), \sigma(2), \cdots \sigma(n)) \in S_n$ if we define

$$P_\sigma(x_{\sigma(1)}) = m(\{x_{\sigma(1)}\})$$

$$P_\sigma(x_{\sigma(2)}) = m(\{x_{\sigma(1)}, x_{\sigma(2)}\}) - m(\{x_{\sigma(1)}\})$$

$$\cdots\cdots\cdots$$

$$P_\sigma(x_{\sigma(i)}) = m(\{x_{\sigma(1)}, \cdots\cdots, x_{\sigma(i)}\}) - m(\{x_{\sigma(1)}, \cdots, x_{\sigma(i-1)}\})$$

$$\cdots\cdots\cdots$$

$$P_\sigma(x_{\sigma(n)}) = 1 - m(\{x_{\sigma(1)}, \cdots\cdots x_{\sigma(n-1)}\})$$

then, obviously, P_σ is a probability distribution on X, we call P_σ the associated probabilities of fuzzy measure m.

Definition 2.2: we will say that two permutations, $\sigma, \sigma^* \in S_n$ are dual if

$$\sigma^*(i) = \sigma(n-i+1), \quad i = 1 \cdots\cdots n$$

Proposition2. 1[1]: A necessary and sufficient condition for two fuzzy measare m and m^* to be dual is that they have the same n! associated probabilities corresponding to dual permutations, that is, $P_\sigma = P^*_{\sigma^*}$ if σ and σ^* are dual.

Proposition 2. 2[1]: A fuzzy measure m is a probability measure if and only if its n! associated probabilities coincide.

Proposition 2.3: Let (m, m^*) be a pair of dual fuzzy measure, if

$$m(A \cup B) + m(A \cap B) > m(A) + m(B), \quad \forall A, B \subseteq X \qquad (2.2)$$

then

$$m^*(A \cup B) + m^*(A \cap B) < m^*(A) + m^*(B) \qquad (2.3)$$

Proof: First, since

$$m^*(A \cup B) = 1 - m(\overline{A \cup B}) = 1 - m(\bar{A} \cap \bar{B}),$$

$$m^*(A \cap B) = 1 - m(\overline{A \cap B}) = 1 - m(\bar{A} \cup \bar{B}).$$

We get:

$$m^*(A\cup B)+m^*(A\cap B)=2-[m(\bar{A}\cap\bar{B})+m(\bar{A}\cup\bar{B})]$$

Besides:

$$m^*(A)+m^*(B)=2-[m(\bar{A})+m(\bar{B})]$$

Frow (2.2), We have:

$$m(\bar{A}\cup\bar{B})+m(\bar{A}\cap\bar{B})>m(\bar{A})+m(\bar{B}).$$

Therefore (2.3) holds.

Proposition 2.4: Let g_λ be a g_λ-measure [2] on $(X, P(X))$ if $\lambda>0$, then

$$g^*_\lambda(A\cup B)+g^*_\lambda(A\cap B)<g^*_\lambda(A)+g^*_\lambda(B) \quad A, B\subseteq X \qquad (2.4)$$

if $\lambda<0$, Then

$$g^*_\lambda(A\cup B)+g^*_\lambda(A\cap B)>g^*_\lambda(A)+g^*_\lambda(B) \qquad (2.5).$$

Where $g^*_\lambda(A)=1-g_\lambda(\bar{A})$.

Proof: From proposition 2.3 and the theorem 2.2 in [3] the (2.4) and (2.5) can be obtained.

Proposition 2.5: Fuzzy measure m is autodual if and only if m is a prob-ability measure.

The proof of proposition 2.5 is obvious.

§3. Entropy of fuzzy measure spaces

Definition3.1: Let $(X, p(X), m)$ is a fuzzy measure space if

$$m(A\cup B)+m(A\cap B)>m(A)+m(B), \quad \forall A, B\subseteq X$$

We will say that

$$\bar{H}(m)=-1/n\sum_{i=1}^{n}\min_{\sigma\in S_n}P_\sigma(x_{(i)})\cdot\min_{\sigma\in S_n}\log[P_\sigma(x_{(i)})] \qquad (3.1)$$

is upper fuzzy entropy of fuzzy measure space $(X, P(X), m)$

If

$$m(A \cup B)+m(A \cap B)<m(A)+m(B)$$

We will say that

$$\underline{H}(m)=-1/n \sum_{i=1}^{n} \max_{\sigma\in S_n} P_{\sigma}(x_i) \cdot \max_{\sigma\in S_n} \log [P_{\sigma}(x_i)] \qquad (3.2)$$

is lower fuzzy entropy of fuzzy measure space $(X, P(X), m)$.

Lemma3. 1: Let m be a fuzzy measure on $(X, P(X))$. Then, for any $A\in P(X)$, there exists a $\sigma_0 \in S_n$, such that

$$P_{\sigma_0}(A) = m(A) \qquad (3.3)$$

proof: For any $B\in P(X)$, let us suppose that

$A\cap B=\{x_{i1}, \cdots, x_{ir}\}$, $\bar{A}\cap B=\{x_{j1}\cdots, x_{js}\}$

$A\cap \bar{B}=\{x_{k1}, \cdots, x_{kt}\}$

We choose a permutation $\sigma_0 \in S_n$, such that

$\sigma_0(1)=i_1, \cdots,$ $\sigma_0(r)=i_r$;

$\sigma_0(r+1)=j_1, \cdots,$ $\sigma_0(r+s)=j_s$

$\sigma_0(r+s+1)=k_1, \cdots,$ $\sigma_0(r+s+t)=k_t$

From the definition of associated probabilities, we have

$$\begin{aligned} P_{\sigma_0}(A) &= \sum_{x\in A} P_{\sigma_0}(x) \\ &= m(\{x_{i1}\})+m(\{x_{i1}, x_{i2}\})-m(\{x_{i1}\})+\cdots\cdots+ \\ &\quad m(\{x_{i1}, \cdots x_{ir}\})-m(\{x_{i1}, \cdots x_{ir-1}\})+ \\ &\quad m(\{x_{i1}, \cdots x_{ir}, x_{j1}, \cdots x_{js}, x_{k1}\})-m(\{x_{i1}, \cdots x_{ir}, x_{j1}, \cdots x_{js}\}) \end{aligned}$$

$$+\cdots+m(\{x_{i1},\cdots x_{ir},x_{j1},\cdots x_{js},x_{k1},\cdots x_{kt}\})$$

$$-m(\{x_{i1},\cdots x_{ir},x_{j1},\cdots x_{js},x_{k1},\cdots x_{kt-1}\})$$

$$=m(\{x_{i1},\cdots x_{ir}\})-m(\{x_{i1},\cdots x_{ir},x_{j1},\cdots x_{js}\})+$$

$$m(\{x_{i1},\cdots x_{ir},x_{j1},\cdots x_{js},x_{k1},\cdots x_{kt}\})$$

$$=m(A\cap B)-m(B)+m(A\cup B)$$

When $B=\emptyset$ we have

$$P_{\sigma_0}(A)=m(A)$$

Proposition 3.1: Let $(X, P(X), m)$ be a fuzzy measure space, if it has upper (lower) fuzzy entropy, then

$$\overline{H}(m)=-1/n\sum_{i=1}^{n} m(x_i)\log(m(x_i)) \tag{3,4}$$

$$(\underline{H}(m)=-1/n\sum_{i=1}^{n} m(x_i)\log(m(x_i)))$$

Proof: We first prove that if

$$m(A\cup B)+m(A\cap B)>m(A)+m(B). \quad \forall A,B\subseteq X$$

then

$$m(A)<\min_{\sigma\in S_n} P_\sigma(A) \qquad \forall A\subseteq X \tag{3.5}$$

suppose that $A=\{x_{i1},x_{i2},\cdots,x_{ir}\}$; then for arbitrary $\sigma\in S_n$

$$P_\sigma(A)=\sum_{j=1}^{r} P_\sigma(x_{ij})=\sum_{j=1}^{r}[m(B_{\sigma_{ij}}\cup\{x_{ij}\})-m(B_{\sigma_{ij}})]$$

Where $B_{\sigma_{ij}}=\{x_{\sigma(1)},x_{\sigma(2)},\cdots,x_{\sigma(hij-1)}\}$, and $\sigma(hij)=ij$

Without losing generality, We can suppose that

$\sigma(i_r) < \sigma(i_{r-1}) < \cdots\cdots < \sigma(i_1)$

If we define the sets A_{ij} by

$A_{ij} = \{ x_{ij}, x_{i,j+1} \cdots\cdots x_{ir} \}, j = 1 \cdots\cdots r,$

then

$$B_{\sigma_{ij}} \cup A_{ij} = B_{\sigma_{ij}} \cup \{ x_{ij} \}$$

$$B_{\sigma_{ij}} \cap A_{ij} = \{ x_{i,j+1}, \cdots\cdots x_{ir} \}, j = 1 \cdots\cdots r-1,$$

$$B_{\sigma_{ir}} \cup A_{ir} = B_{\sigma_{ir}} \cup \{ x_{ir} \}, B_{\sigma_{ir}} \cap A_{ir} = \phi$$

Thus

$$\sum_{j=1}^{r} m(B_{\sigma_{ij}} \cup \{x_{ij}\}) > \sum_{j=1}^{r} m(B_{\sigma_{ij}}) + \sum_{j=1}^{r} m(A_{ij}) - \sum_{j=1}^{r-1} m(A_{i,j+1})$$

$$= \sum_{j=1}^{r} m(B_{\sigma_{ij}}) + \sum_{j=1}^{r} m(A_{ij}) - \sum_{j=2}^{r} m(A_{ij})$$

$$= \sum_{j=1}^{r} m(B_{\sigma_{ij}}) + m(A_{i1})$$

Therefore

$$m(A_{i1}) = m(A) < \sum_{j=1}^{r} [m(B_{\sigma_{ij}} \cup \{x_{ij}\}) - m(B_{\sigma_{ij}})] = P_{\sigma}(A)$$

Hence

$$m(A) < \min_{\sigma \in S_n} P_{\sigma}(A) \qquad \forall A \subseteq X$$

By lemma 3.1, we know that there always exists a $\sigma \in S_n$ such that $m(A) = P_{\sigma}(A)$, it is obvious that

$$m(A) = \min_{\sigma \in S_n} P_{\sigma}(A). \qquad \forall A \subseteq X$$

Therefore,

$$\overline{H}(m) = -1/n \sum_{i=1}^{n} \min_{\sigma \in S_n} P_{\sigma}(x_i) . \min_{\sigma \in S_n} \log[P_{\sigma}(x_i)]$$

$$= -1/n \sum_{i=1}^{n} m(x_i) \log \min_{\sigma \in S_n} (m(x_i)) = -1/n \sum_{i=1}^{n} m(x_i) \log m(x_i).$$

Analogously if

$$m(A \cup B) + m(A \cap B) < m(A) + m(B), \quad \forall A, B \subseteq X$$

we have

$$\underline{H}(m) = -1/n \sum_{i=1}^{n} m(x_i) \log(m(x_i))$$

Henc the (3.4) holds.

Propsition3.2: Let fuzzy measure m be autodual, then $\overline{H}(m) = \underline{H}(m)$

Proof: This is evident from proposition 2.5.

Proposition3.4: Let b be a belief measure on $(X, P(X))$, then $\overline{H}(b) < \underline{H}(L)$.

Where L is a plausibility measure on $(X, P(X))$.

Proof: obviously.

Proposition3.5: Let Π be a possibility measure on $(X, P(X))$, then $\overline{H}(\Pi) < \underline{H}(T)$. Where T is a necessity measure [3] on $(X, P(X))$.

Proof: obviously.

§4. Entropy of information space.

Let $X = \{x_1 \cdots\cdots x_n\}$ be a universe of discourse, $\Omega = \{\omega_1 \cdots\cdots \omega_m\}$ is a set of the characters of X. We denote

$$X \times \Omega = \{(x, \omega): x \in X, \omega \in \Omega\}$$

Definition 4.1: $I \subset X \times \Omega$ is called a relation from X to Ω, if it satisfies the following conditions:

(i) $\forall x \in X$, $\exists \omega \in \Omega$, shch that $(x, \omega) \in I$.

(ii) $\forall \omega \in \Omega, \exists x \in X$, such that $(x, \omega) \in I$.

(iii) $\exists$ $i < n$, $j, k < m (j \neq k)$, when $(x_i, \omega_j) \in I$, $(x_i, \omega_k) \notin I$.

For ang $A \subset X$, $\alpha \subset \Omega$, we set

$$f(A) = \{\omega : \forall x \in A, (x, \omega) \in I\}$$

$$g(\alpha) = \{x : \forall \omega \in \alpha, (x, \omega) \in I\}$$

Definition4. 2: For $A \subset X$, $\alpha \subset \Omega$, (A, α) is called a conception, if $f(A) = \alpha$, $g(\alpha) = A$,

Definition4.3: Let (A, α) be a conception, we will call

$$I(\alpha) = \Pi\{\overline{g(\alpha)}\} = \Pi\{\overline{A}\}$$

an informative quantity of conception (A, α). Where Π is a possibility measures generated by a possibility distribution on X.

Proposition4.1: The informative quantity of conception (A, α) is a fuzzy measure.

Proof: Frist, it is obvious that $g(\phi)=X, g(\Omega)=\emptyset$, and therefore we have.

$I(\phi)=\Pi\{\overline{g(\phi)}\}=\Pi(\phi)=0$

$I(\Omega)=\Pi\{\overline{g(\Omega)}\}=\Pi(X)=1$

Next let $\alpha_1, \alpha_2 \subset \Omega, \alpha_1 \subset \alpha_2$, if $x \in g(\alpha_2)$, then $\forall \omega \in \alpha_2, (x,\omega) \in I$.

Since $\alpha_1 \subset \alpha_2$, we have $\forall \omega \in \alpha_1, (x,\omega) \in I$, that is, $g(\alpha_1) \supset g(\alpha_2)$, therefore $\overline{g(\alpha_1)} \subset \overline{g(\alpha_2)}$ and $I(\alpha_1) < I(\alpha_2)$.

Definition 4.4: $H(I(\alpha)) = -\sum_{\omega_i \in \alpha} I(\omega_i) \log(I(\omega_i))$

Will be called entropy of conception (A,α).

Definition 4.5: conceptions (A,α) and (B,β) are called independent, if and only if $I(\alpha \cap \beta) = I(\alpha) I(\beta)$.

Proposition 4.2: If (A,α) and (B,β) are independent,

$\sum_{\omega \in \alpha} I(\omega)=1, \sum_{\omega \in \beta} I(\omega)=1$, then

$$H(I(\alpha\cap\beta)) = H(I(\alpha)) + H(I(\beta)),$$

Proof: From definition 4.4 and 4.5 we have

$$H(I(\alpha\cap\beta)) = -\sum_{\substack{\omega_k \in \alpha \\ \omega_l \in \beta}} I(\alpha) I(\beta) \log [I(\alpha) I(\beta)]$$

$$= -\sum_{\substack{\omega_k \in \alpha \\ \omega_l \in \beta}} I(\alpha) I(\beta) \log(I(\alpha) - \sum_{\substack{\omega_k \in \alpha \\ \omega_l \in \beta}} I(\alpha) I(\beta) \log[I(\beta)]$$

$$= -\sum_{\omega_k \in \alpha} I(\alpha) \log(I(\alpha) - \sum_{\omega_l \in \beta} I(\beta) \log[I(\beta)]$$

$$= H(I(\alpha)) + H(I(\beta))$$

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