Multicriteria fuzzy linear programming with Yager's parametrized t-norm

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Abstract: This paper proposes to use Yager's parameterised t-norm T_p to solve multicriteria fuzzy linear optimisation problems. Considering flat fuzzy numbers with linear reference functions the T_p-based additions in objectives and/or in left-hand sides of constraints lead to flat fuzzy numbers having linear reference functions as well (independently of the value of parameter p). Varying the parameter p the decision maker can choose different aggregation concepts for particular constraints or objectives between the extremes of pessimistic $T_p = T_\infty = \min$ and optimistic $T_p(x,y) = T_1(x,y) = \max\{0, x+y-1\}$ approaches and simultaneously control the growth of spreads i. e. the growth of uncertainty in calculations.

Keywords: Multicriteria fuzzy linear optimisation, extension principle, triangular norm, flat fuzzy intervals.

1. Introduction

Let us consider a multicriteria LP-model of the form

$$\left.\begin{array}{c} c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n \\ \vdots \qquad \vdots \qquad \qquad \vdots \\ c_{r1}x_1 + c_{r2}x_2 + \ldots + c_{rn}x_n \end{array}\right\} \longrightarrow \max$$

subject to (1)

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \le b_i \qquad i = 1, \ldots, m$$

$$x_j \ge 0 \qquad j = 1, \ldots, n$$

Solving real decision problems, we often encounter the difficulty that the parameters a_{ij}, c_{kj}, $b_i \in \mathbb{R}$ are not known exactly. A suitable way to model these imprecise data is

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to use fuzzy sets. Replacing the crisp parameters by fuzzy sets $\tilde{A}_{ij}$, $\tilde{C}_{kj}$ and $\tilde{B}_i$ we get a multicriteria fuzzy LP-model (2).

Following the methods described in Rommelfanger [8,10] we assume that a compromise solution of (1) is got by an interactive process which is based on fuzzy aspiration levels $\tilde{D}_k$ specified for each of the goals.

In doing so, the task at each iteration step is to find a solution of the system of fuzzy constraints

$$\tilde{C}_{k1}x_1 + \tilde{C}_{k2}x_2 + \ldots + \tilde{C}_{kn}x_n \tilde{\gtrsim} \tilde{D}_k \qquad k = 1, \ldots, r \tag{2}$$

$$\tilde{A}_{i1}x_1 + \tilde{A}_{i2}x_2 + \ldots + \tilde{A}_{in}x_n \tilde{\lesssim} \tilde{B}_i \qquad i = 1, \ldots, m \tag{3}$$

for $x_j \geq 0$, $j = 1, \ldots, n$ where $\tilde{D}_k$ are the aspiration levels specified by decision maker.

The majority of approaches in the literature use Zadeh's extension principle based on the minimum-norm for summarizing the left sides of the constraints (2), (3).

Dubois and Prade [1] showed that if all the coefficients $\tilde{C}_{kj} = (c_{kj}^L, c_{kj}^R, \gamma_{kj}^L, \gamma_{kj}^R)_{LR}$ or $\tilde{A}_{ij} = (a_{ij}^L, a_{ij}^R, \alpha_{ij}^L, \alpha_{ij}^R)_{LR}$, $j = 1 \ldots n$ are (flat) fuzzy numbers of the same LR-type than the left sides of (2), (3) may be summed up to LR-fuzzy intervals

$$\tilde{C}_k(x_1, \ldots, x_n) = (c_k^L, c_k^R, \gamma_k^L, \gamma_k^R)_{LR} \tag{4k}$$

where

$$c_k^L = \sum_{j=1}^n c_{kj}^L x_j \qquad c_k^U = \sum_{j=1}^n c_{kj}^U x_j \qquad \gamma_k^L = \sum_{j=1}^n \gamma_{kj}^L x_j \qquad \gamma_k^U = \sum_{j=1}^n \gamma_{kj}^U x_j \tag{5k}$$

or

$$\tilde{A}_i(x_1, \ldots, x_n) = (a_i^L, a_i^U, \alpha_i^L, \alpha_i^U)_{LR} \tag{6i}$$

where

$$a_i^L = \sum_{j=1}^n a_{ij}^L x_j \qquad a_i^U = \sum_{j=1}^n a_{ij}^U x_j \qquad \alpha_i^L = \sum_{j=1}^n \alpha_{ij}^L x_j \qquad \alpha_i^U = \sum_{j=1}^n \alpha_{ij}^U x_j \tag{7i}$$

In the literature, there exist different concepts for interpreting the inequality relation in fuzzy constraints

$$\tilde{C}_k \gtrsim \tilde{D}_k \qquad k = 1, \ldots, r \tag{8}$$

$$\tilde{A}_i \lesssim \tilde{B}_i \qquad i = 1, \ldots, m \tag{9}$$

Well known interpretation of the inequality relation $\lesssim$ in fuzzy constraints of type (9) are those of Negoita, Sularia [5], Tanaka, Asai [12], Ramík and Rímanek [7], Slowinski [11] and Rommelfanger [8,10]; see the detailed survey in [9].

In this paper we adopt from [10] the very flexible interpretation "$\lesssim_R$":

$$\tilde{A}_i(\mathrm{x}) \lesssim_R \tilde{B}_i \quad \Longleftrightarrow \quad \begin{cases} \sum_{j=1}^{n} \left(a_{ij}^U + \alpha_{ij}^U R^{-1}(\epsilon)\right) x_j \leq b_i + \beta_i R^{-1}(\epsilon) & (10) \\ \mu_{H_i}\left(\sum_{j=1}^{n} a_{ij}^U x_j\right) \rightarrow \max & (11) \end{cases}$$

where

$$\mu_{H_i}(y) = \begin{cases} 1 & \text{if } y < b_i \\ \mu_{B_i}(y) & \text{if } y \geq b_i \end{cases} \tag{12}$$

and μ_{B_i} is the membership function of the right-hand side of the LR-fuzzy number $\tilde{B}_i = \left(b_i, \beta_i^L, \beta_i^U\right)_{LR}$; see Figure 1.

Slowinski [11] called the inequality (10) the *pessimistic index* in comparison with another inequality which was named as *optimistic index*. But, we think the denotation *pessimistic* is correct, because the summarizing of the left sides of fuzzy constraints (3) is based on the pessimistic minimum-operator so that the spreads $\alpha_i^U = \sum_{j=1}^{n} \alpha_{ij}^U x_j$ increase largely with the number and size of the variables x_j.

In this paper we present a new, more flexible alternative for summarizing the left sides of the constraints (3), when the extended addition is based on Yager's parametrized t-norm T_p. Varying the parameter p the decision maker can choose different aggregation concepts between the pessimistic minimum-operator and the optimistic Lukasiewicz's t-norm $T_1(x, y) = T_L(x, y) = \max\{0, x + y - 1\}$.

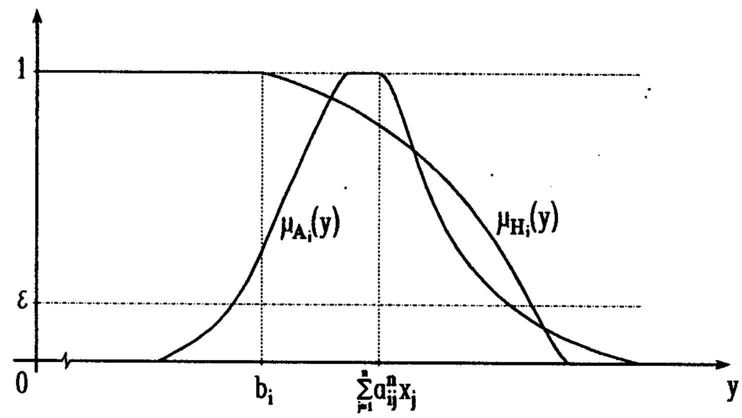

Figure 1.

2. Modelling the fuzzy parameters

For simplifying the formulas we remark that in practice all the (flat) fuzzy numbers in (3) may be described by piecewise linear membership functions. A practical way of modelling suitable membership functions is proposed in [10]. At first the decision maker has to specify some prominent membership levels. E.g.

$\alpha = 1 \;:\; \mu_{B_i}(y) = 1$ means that the value y with certainty belongs to the set of available values,

$\alpha = \lambda_A \;:\; \mu_{B_i}(y) \geq \lambda_A$ means that the decision maker (DM) is willing to accept y as an available value for the time being. A value y with $\mu_{B_i}(y) \geq \lambda_A$ has a good chance of belonging to the set of available values. Corresponding values of y are relevant to the decision. Obviously, a value y with $\mu_{B_i}(y) = \lambda_A$ is a sort of aspiration level.

$\alpha = \epsilon \;:\; \mu_{B_i}(y) < \epsilon$ means that y has only a very little chance of belonging to the set of available values. The DM is willing to neglect the values with $\mu_{B_i}(y) < \epsilon$.

Subsequently the decision maker has to fix values $b_i^{\lambda_A}$ and b_i^{ϵ} such that $\mu_{B_i}(b_i^{\lambda_A}) = \lambda_A$ and

$\mu_{B_i}(b_i^\epsilon) = \epsilon$.

Then the polygon line from $(b_i, 1)$ over $(b_i^{\lambda_A}, \lambda_A)$ to (b_i^ϵ, ϵ) is a suitable approach to μ_{B_i} on the interval $[b_i, b_i^\epsilon]$. For all $y \notin [b_i, b_i^\epsilon]$ we set $\mu_{B_i}(y) = 0$; see Figure 2. Taking pattern from LR-type fuzzy numbers we symbolize a fuzzy number with this special membership function by $\tilde{B}_i = \left(b_i; 0, 0; \beta_i^{\lambda_A}, \beta_i^\epsilon\right)^{\lambda_A, \epsilon}$, where

$$\beta_i^{\lambda_A} = b_i^{\lambda_A} - b_i \qquad \text{and} \qquad \beta_i^\epsilon = b_i^\epsilon - b_i$$

If required, the decision maker may specify some additional membership levels and additional points $(y, \mu_{B_i}(y))$ on the polygon line.

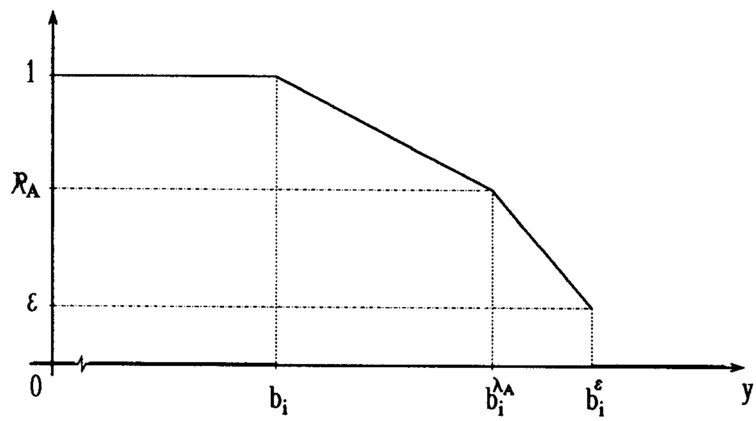

Figure 2.

On the analogy, the fuzzy aspiration level $\tilde{D}_k$ can be described by

$$\tilde{D}_k = \left(d_k; \delta_k^{\lambda_A}, \delta_k^\epsilon; 0, 0\right)^{\lambda_A, \epsilon}$$

The coefficients $\tilde{A}_{ij}$ and $\tilde{C}_{kj}$ may be characterized by trapezoid fuzzy intervals. As the spreads of the coefficients are much smaller then those of the right sides, we will neglect the level λ_A and use the representation

$$\tilde{A}_{ij} = \left(a_{ij}^L, a_{ij}^U, \alpha_{ij}^{L\epsilon}, \alpha_{ij}^{U\epsilon}\right)^\epsilon \qquad \text{and} \qquad \tilde{C}_{kj} = \left(c_{kj}^L, c_{kj}^U, \gamma_{kj}^{L\epsilon}, \gamma_{kj}^{U\epsilon}\right)^\epsilon$$

3. t-norm based addition

A real operation $* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ can be extended to fuzzy sets on $\mathbb{R}$ in the sense of Zadeh's extension principle as follows

$$\mu_{\tilde{a}*\tilde{b}}(z) = \sup_{x*y=z} T\big(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)\big) \qquad z \in \mathbb{R} \tag{13}$$

where $\tilde{a}$, $\tilde{b} \in \mathcal{F}(\mathbb{R})$ and $T : [0,1] \times [0,1] \to [0,1]$ is an arbitrary t-norm. As it was mentioned above, linear optimization methods known from the literature use mostly the minimum-norm ($T = \min$).

For getting a more flexible concept, now we propose to define the addition via the extension principle based on a t-norm having zero divisor, namely, on Yager's parametrized t-norm:

$$T_p(x,y) = \max\left\{0, 1 - \big((1-x)^p + (1-y)^p\big)^{1/p}\right\} \qquad x, y \in [0,1], \quad p > 0 \tag{14}$$

or in case of n variables

$$T_p(t_1, \ldots, t_n) = \max\left\{0, 1 - \left(\sum_{i=1}^{n}(1-t_i)^p\right)^{1/p}\right\} \qquad t_1, \ldots, t_n \in [0,1] \tag{15}$$

For the special case, that all the coefficients $\tilde{A}_{ij}$ are trapezoid fuzzy intervals of type $\tilde{A}_{ij}(x) = \left(a_{ij}^L, a_{ij}^U, \alpha_{ij}^L, \alpha_{ij}^U\right)$ there exist the following theorem:

Theorem 1. Suppose the coefficients $\tilde{A}_{ij}$ of the left hand side of the inequality constraints

$$\tilde{A}_{i1}x_1 + \tilde{A}_{i2}x_2 + \ldots + \tilde{A}_{in}x_n \tilde{\lesssim} \tilde{B}_i \qquad i = 1, \ldots, m \tag{16i}$$

are trapezoid fuzzy intervals of type $\tilde{A}_{ij}(x) = \left(a_{ij}^L, a_{ij}^U, \alpha_{ij}^L, \alpha_{ij}^U\right)$. If the addition is extended by Yager's t-norm T_p (14) with $p \geq 1$ then

$$\tilde{A}_i(\mathrm{x}) = \tilde{A}_{i1}x_1 + \tilde{A}_{i2}x_2 + \ldots + \tilde{A}_{in}x_n = \left(a_i^L(\mathrm{x}), a_i^U(\mathrm{x}), \alpha_i^L(p,\mathrm{x}), \alpha_i^U(p,\mathrm{x})\right) \tag{17}$$

is also a fuzzy interval with linear reference functions and

$$a_i^L(\mathbf{x}) = \sum_{j=1}^{n} a_{ij}^L x_j \qquad a_i^U(\mathbf{x}) = \sum_{j=1}^{n} a_{ij}^U x_j$$

$$\alpha_i^L(p,\mathbf{x}) = \left\|\left(x_1\alpha_{i1}^L,\ldots,x_n\alpha_{in}^L\right)\right\|_q = \left(\left(x_1\alpha_{i1}^L\right)^q + \ldots + \left(x_n\alpha_{in}^L\right)^q\right)^{1/q} \tag{18}$$

$$\alpha_i^U(p,\mathbf{x}) = \left\|\left(x_1\alpha_{i1}^U,\ldots,x_n\alpha_{in}^U\right)\right\|_q = \left(\left(x_1\alpha_{i1}^U\right)^q + \ldots + \left(x_n\alpha_{in}^U\right)^q\right)^{1/q}$$

where $q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 1. The proof is based on results of Fullér, Keresztfalvi [2] and on the following fact originally used by Kovács [3,4]: there exists a natural bijection between the normed linear space $\mathbb{R}^n$ and the normed linear space of linear functionals on $\mathbb{R}^n$, i.e. $Lin(\mathbb{R}^n, \mathbb{R})$. Moreover, this bijection is an isometry if we normalize $Lin(\mathbb{R}^n, \mathbb{R})$ with the $\|\cdot\|_p$ norm and $\mathbb{R}$ with the $\|\cdot\|_q$ norm where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 2. We can see that $\alpha_i^L(p,\mathbf{x})$ and $\alpha_i^U(p,\mathbf{x})$ decrease while p decreases (q increases). This property provides a good possibility of controlling the growth of spreads i.e. the growth of uncertainty during the calculations. From (20) it follows that using T_p-based addition we neglect the combinations

$$\left\{(t_1,\ldots,t_n) \;\middle|\; T_p\left(\mu_{A_{i1}}(t_1),\ldots,\mu_{A_{in}}(t_n)\right) = 0\right\} \tag{21}$$

Decreasing the parameter p the zero divisor of T_p becomes greater, that is, the set of neglected combinations and the risk taken by decision maker increases.

Both extreme cases are worth mentioning:

(i) If $p = 1$ (and $q = \infty$) then $T_p = T_L$ is the well known Lukasiewicz's t-norm: $T_1(x,y) = T_L(x,y) = \max\{0, x+y-1\}$. In this case $\tilde{A}_i(\mathbf{x})$ have the smallest spreads

$$\alpha_i^L(1,\mathbf{x}) = \max\{x_1\alpha_{i1}^L,\ldots,x_n\alpha_{in}^L\}$$

$$\alpha_i^U(1,\mathbf{x}) = \max\{x_1\alpha_{i1}^U,\ldots,x_n\alpha_{in}^U\}$$

but the decision maker takes the greatest risk.

(ii) It is also easy to see that if p tends to infinity then T_p tends to the min-norm and we come back to the min-based addition. Thus, if $q = 1$ (and $p = \infty$) then $\tilde{A}_i(\mathbf{x})$ have the greatest spreads:

$$\alpha_i^L(\infty, \mathbf{x}) = x_1\alpha_{i1}^L + \ldots + x_n\alpha_{in}^L$$

$$\alpha_i^U(\infty, \mathbf{x}) = x_1\alpha_{i1}^U + \ldots + x_n\alpha_{in}^U$$

and there are no neglected values in (21) with positive grade of membership.

On the other hand, the 1-level set $[a_i^L(\mathbf{x}), a_i^U(\mathbf{x})]$ does not change with the parameter p.

4. Optimization process

Using the extended addition based on the t-norm T_p we have an opportunity to change the inequality relation $\lesssim_R$ to

$$\tilde{A}_i(\mathbf{x}) \lesssim_{KR} \tilde{B}_i \quad \Longleftrightarrow \quad \begin{cases} a_i^U(\mathbf{x}) + \alpha_i^U(p, \mathbf{x})(1-\epsilon) \leq b_i + \beta_i^\epsilon & (22) \\ \mu_{H_i}(a_i^U(\mathbf{x})) \to \max & (11) \end{cases}$$

For linear reference function R and for $p = \infty$ this version is identical with the original form, but in general this inequality can be varied with p.

Now, the decision maker has two ways to express his risk mentality:

(*i*) by specifying the values a_{ij}^ϵ, c_{kj}^ϵ, b_i^ϵ, d_k^ϵ;

(*ii*) by choosing a value $p \in [1, +\infty[$ for each objective and for each constraint independently.

In order to demonstrate the influence of p, we consider the following simple example:

$$(2, 3; 1; 1)^\epsilon x_1 + (4, 5; 1.5; 2)^\epsilon x_2 \lesssim (30; 0; 6)^\epsilon$$

(*i*) using the min-operator we get according to (10)

$$4x_1 + 7x_2 \leq 36$$

(see the set of feasible solutions X_∞ in Figure 3.)

(*ii*) **using the other extreme, Lukasiewicz's t-norm (i.e. $p = 1$) we get**

$$3x_1 + 5x_2 + \max\{x_1, 2x_2\} \leq 36$$

which is equivalent to the system

$$\begin{cases} 4x_1 + 5x_2 \leq 36 \\ 3x_1 + 7x_2 \leq 36 \end{cases} \tag{23}$$

The set of feasible solutions of (23) is by the set M_1 greater than X_∞, i.e. equal to $X_1 = X_\infty \cup M_1$. If the number and the size of the variables x_j increase, the part of M_1 in X_1 increase too.

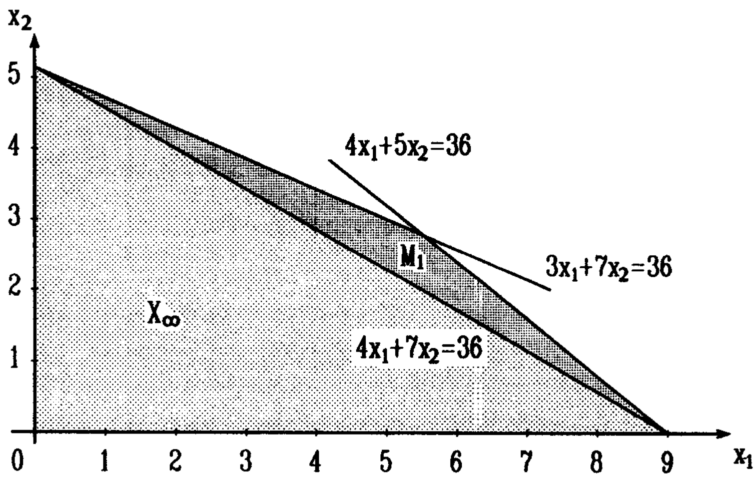

Figure 3.

Returning to the system of fuzzy inequalities (2), (3), we model the fuzzy aspiration level $\tilde{D}_k$ in analogy to $\tilde{B}_i$ as $\tilde{D}_k = \left(d_k; \delta_k^{\lambda_A}, \delta_k^{\epsilon}\right)^{\lambda_A, \epsilon}$ and introduce for the extended addition of the left sides $\tilde{C}_k(\mathbf{x}) = \tilde{C}_{k1}x_1 + \tilde{C}_{k2}x_2 + \ldots + \tilde{C}_{kn}x_n$, based on t-norm T_p the corresponding notations

$$c_k^L(\mathbf{x}) = \sum_{j=1}^{n} c_{kj}^L x_j \qquad\qquad c_k^U(\mathbf{x}) = \sum_{j=1}^{n} c_{kj}^U x_j$$

$$\gamma_k^L(p, \mathbf{x}) = \left\| \left(x_1\gamma_{k1}^L, \ldots, x_n\gamma_{kn}^L\right) \right\|_q = \left(\left(x_1\gamma_{k1}^L\right)^q + \ldots + \left(x_n\gamma_{kn}^L\right)^q \right)^{1/q} \tag{24}$$

$$\gamma_k^U(p, \mathbf{x}) = \left\| \left(x_1\gamma_{k1}^U, \ldots, x_n\gamma_{kn}^U\right) \right\|_q = \left(\left(x_1\gamma_{k1}^U\right)^q + \ldots + \left(x_n\gamma_{kn}^U\right)^q \right)^{1/q}$$

Using the inequality relation $\lesssim_{KR}$ we have

$$\tilde{D}_k \lesssim_{KR} \tilde{C}_k(\mathbf{x}) \quad \Longleftrightarrow \quad \begin{cases} c_k^L(\mathbf{x}) - \gamma_k^L(p,\mathbf{x})(1-\epsilon) \geq d_k - \delta_i^\epsilon & (25) \\ \mu_{Z_k}\left(c_k^L(\mathbf{x})\right) \rightarrow \max & (26) \end{cases}$$

where

$$\mu_{Z_k}(y) = \begin{cases} 1 & \text{if } y > d_k \\ \mu_{D_k}(y) & \text{if } d_k - \delta_k^\epsilon \leq y \leq d_k \\ 0 & \text{if } y < d_k - \delta_k^\epsilon \end{cases} \tag{27}$$

Therefore, the system (2), (3) is equivalent to the crisp optimization problem

$$\left(\mu_{Z_1}\left(c_1^L(\mathbf{x})\right), \ldots, \mu_{Z_r}\left(c_r^L(\mathbf{x})\right), \mu_{H_1}\left(a_1^U(\mathbf{x})\right), \ldots, \mu_{H_m}\left(a_m^U(\mathbf{x})\right)\right) \longrightarrow \max \tag{28}$$

subject to

$$\begin{cases} -c_k^L(\mathbf{x}) + \gamma_k^L(p,\mathbf{x})(1-\epsilon) \leq -d_k + \delta_i^\epsilon & \text{k=1,...,r} \\ a_i^U(\mathbf{x}) + \alpha_i^U(p,\mathbf{x})(1-\epsilon) \leq b_i + \beta_i^\epsilon & \text{i=1,...,m} \\ x_j \geq 0 & \text{j=1,...,n} \end{cases} \tag{29}$$

Denoting by X_p the set of feasible solutions of the constraint system (29), we have

$$X_\infty \subset X_{p_2} \subset X_{p_1} \subset X_1 \qquad \text{if} \qquad 1 < p_1 < p_2 < \infty$$

because the spreads $\alpha_i^U(p,\mathbf{x})$ and $\gamma_k^L(p,\mathbf{x})$ are decreasing functions of $p \in [1, +\infty]$.

In particular, we can discriminate the following cases:

(i) if $p = \infty$, the constraint system (29) consist of $r + m$ linear inequalities and n non-negative-restrictions; the set of feasible solutions X_∞ is a simplex.

(ii) if $p = 1$, we get instead of each of the inequalities in (29) n linear inequalities, so that X_1 is a simplex too.

(iii) if $p \in]1, +\infty[$ we have

$$\alpha_i^L(p,\mathbf{x}) = \left\|\left(x_1\alpha_{i1}^L, \ldots, x_n\alpha_{in}^L\right)\right\|_q = \left(\left(x_1\alpha_{i1}^L\right)^q + \ldots + \left(x_n\alpha_{in}^L\right)^q\right)^{1/q} \tag{30}$$

$$\alpha_i^U(p,\mathbf{x}) = \left\|\left(x_1\alpha_{i1}^U,\ldots,x_n\alpha_{in}^U\right)\right\|_q = \left(\left(x_1\alpha_{i1}^U\right)^q + \ldots + \left(x_n\alpha_{in}^U\right)^q\right)^{1/q}$$

$$\gamma_k^L(p,\mathbf{x}) = \left\|\left(x_1\gamma_{k1}^L,\ldots,x_n\gamma_{kn}^L\right)\right\|_q = \left(\left(x_1\gamma_{k1}^L\right)^q + \ldots + \left(x_n\gamma_{kn}^L\right)^q\right)^{1/q} \quad (31)$$

$$\gamma_k^U(p,\mathbf{x}) = \left\|\left(x_1\gamma_{k1}^U,\ldots,x_n\gamma_{kn}^U\right)\right\|_q = \left(\left(x_1\gamma_{k1}^U\right)^q + \ldots + \left(x_n\gamma_{kn}^U\right)^q\right)^{1/q}$$

with $q = \frac{p}{p-1}$. In these cases the inequalities in (29) can not be replaced by linear ones, but the set of feasible solutions X_p is convex for all $p \in]1,+\infty[$. Convexity of X_p can easily be proved using the triangle inequality for $\|\cdot\|_p$.

As to practical problems, a decision maker is not interested in getting the complete solution of the system (28), (29), but he needs a procedure which generates a so-called compromise solution.

In order to determine a compromise solution, we introduce Zadeh's minimum-operator as an appropriate preference function. Obviously, the compromise objective function

$$\Lambda(\mathbf{x}) = \min\left\{\mu_{Z_1}\left(c_1^L(\mathbf{x})\right),\ldots,\mu_{Z_r}\left(c_r^L(\mathbf{x})\right),\mu_{H_1}\left(a_1^U(\mathbf{x})\right),\ldots,\mu_{H_m}\left(a_m^U(\mathbf{x})\right)\right\} \quad (32)$$

expresses the total satisfaction of the decision maker on X_p.

If X_p is not empty, the decision problem

$$\max_{x\in X_p} \min\left\{\mu_{Z_1}\left(c_1^L(\mathbf{x})\right),\ldots,\mu_{Z_r}\left(c_r^L(\mathbf{x})\right),\mu_{H_1}\left(a_1^U(\mathbf{x})\right),\ldots,\mu_{H_m}\left(a_m^U(\mathbf{x})\right)\right\}$$

is equivalent to the optimization problem

$$\Lambda \longrightarrow \max$$

subject to (33)

$$\begin{aligned}
&\Lambda \le \mu_{Z_k}\left(c_k^L(\mathbf{x})\right) && k = 1,\ldots,r\\
&\Lambda \le \mu_{H_i}\left(a_i^U(\mathbf{x})\right) && i = 1,\ldots,m\\
&\mathbf{x} \in X_p\ , \quad 0 \le \Lambda \le 1
\end{aligned}$$

Following the explanation in [10], we adopt that all the piecewise linear functions μ_{Z_k} and μ_{H_i} are concave membership functions. Then, if $p = 1$ or $p = \infty$, the system (33) is a crisp LP-problem and a solution can be got by using the well-known simplex procedures.

But, for $p \in]1, +\infty[$ we arrive at a non-linear problem. If X_p is convex then the set of feasible sulotions of the system (33) is convex too and there exists an optimal solution of (33), see. e.g. Neumann [6]. For getting a compromise solution one of the reduced gradient methods can be applied.

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