On t-norms which are related to distances of fuzzy sets

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Abstract

If one considers the membership degrees of fuzzy sets as truth values of a many-valued logic there is a canonical way to define a fuzzified equality for fuzzy sets. The definitions contain as parameters t-norms which act as truth functions of the conjunction connective. For some such t-norms the negation of the fuzzified equality has the property of a metric. The paper gives a necessary and sufficient condition which characterizes all those t-norms which in that way yield a metric for fuzzy sets.

1 The basic notions

We assume that a fixed universe of discourse $\mathcal{X}$ is given which contains at least two elements. The membership degrees $\mu_A(x)$ are considered as truth degrees of a generalized, i.e. many-valued membership predicate ε. For membership of a point $a \in \mathcal{X}$ in a fuzzy set $A \in \mathbb{F}(\mathcal{X})$ we then write: $a \,\varepsilon\, A$. But, as usual in formal logic, now we have to distinguish between this well-formed formula and its truth degree $[\![a \,\varepsilon\, A]\!]$ which is nothing else than the usual membership degree: $[\![a \,\varepsilon\, A]\!] =_{def} \mu_A(a)$.

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This notation $[\![\ldots]\!]$ for the truth degree will also be used in case ... is a more complex expression. As basic tools to built up more complex expressions, connectives for conjunction, implication, negation and a quantifier for generalization will be used. As usual, the negation operator $\neg$ is characterized by

$$[\![\neg H]\!] =_{def} 1 - [\![H]\!] \tag{1}$$

if H is any well-formed formula of the (set theoretic) language we just are constituting. Quite standard, too, is the understanding of the generalization quantifier $\forall$ which is read as the infimum of the corresponding truth degrees:

$$[\![\forall x\, H(x)]\!] =_{def} \inf_{x \in \mathcal{X}} [\![H(x)]\!]. \tag{2}$$

A wide variety of possibilities exists to interpret the conjunction connective $\wedge$. We will allow any t-norm, cf. e. g. [1], [6], [7], [8], to be used as truth function for $\wedge$. And we write $\wedge_{\boldsymbol{t}}$ to indicate, that $\boldsymbol{t}$ is the truth function which characterizes $\wedge_{\boldsymbol{t}}$. Hence one always has

$$[\![H_1 \wedge_{\boldsymbol{t}} H_2]\!] =_{def} [\![H_1]\!]\ \boldsymbol{t}\ [\![H_2]\!]. \tag{3}$$

There is a special case, the so called Łukasiewicz conjunction & characterized via (3) using the t-norm

$$\boldsymbol{t}_{\text{Luk}}(u,v) =_{def} \max\{0, u+v-1\}. \tag{4}$$

For t-norms we write $\boldsymbol{t}_1 \leqq \boldsymbol{t}_2$ iff $\boldsymbol{t}_1(u,v) \le \boldsymbol{t}_2(u,v)$ for all $u,v \in [0,1]$.

Among the t-norms the *left-continuous* ones are of special interest. With them by the definition

$$u\, \varphi_{\boldsymbol{t}}\, v =_{def} \sup\{w \mid u\, \boldsymbol{t}\, w \le v\} \quad \text{for all } u,v \in [0,1] \tag{5}$$

a Φ-operator $\varphi_{\boldsymbol{t}}$ is connected which is the truth function of a suitable implication connective $\rightarrow_{\boldsymbol{t}}$ to be considered together with $\wedge_{\boldsymbol{t}}$; cf. [2], [3]. (For the t-norm $\boldsymbol{t}_{\text{Luk}}$ the corresponding Φ-operator φ_{Luk} is the well known Łukasiewicz implication: $u\, \varphi_{\text{Luk}}\, v = \min\{1, 1-u+v\}$.) The left continuous t-norms $\boldsymbol{t}$ also have another important property:

$$u\, \varphi_{\boldsymbol{t}}\, v = 1 \;\Leftrightarrow\; u \le v \quad \text{for all } u,v \in [0,1] \tag{6}$$

which is essential for the proof of the theorem.

One of the basic relations for fuzzy sets $A, B \in \mathbb{F}(\mathcal{X})$ over a given universe of discourse $\mathcal{X}$ is their inclusion relation

$$A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x) \quad \text{for all } x \in \mathcal{X}. \tag{7}$$

With the logical preliminaries the fuzzified inclusion now can be defined by

$$A \mathrel{\underline{\underline{\subset}}}_{\boldsymbol{t}} B =_{def} \forall x(x \,\varepsilon\, A \rightarrow_{\boldsymbol{t}} x \,\varepsilon\, B), \tag{8}$$

which means in more traditional notation

$$[\![A \mathrel{\underline{\underline{\subset}}}_{\boldsymbol{t}} B]\!] = \inf_{x \in \mathcal{X}} \sup\{w \mid [\![x \,\varepsilon\, A]\!] \,\boldsymbol{t}\, w \leq [\![x \,\varepsilon\, B]\!]\}. \tag{9}$$

Obviously, this is a direct generalization of (7). For, if $\mu_A(x) \leq \mu_B(x)$ one always has $\mu_A(x) \,\varphi_{\boldsymbol{t}}\, \mu_B(x) = 1$, therefore from $A \subseteq B$ one gets $[\![A \mathrel{\underline{\underline{\subset}}}_{\boldsymbol{t}} B]\!] = 1$. And conversely, if $\mu_A(x) \,\varphi_{\boldsymbol{t}}\, \mu_B(x) = 1$ then for finite $\mathcal{X}$ one immediately gets $\mu_A(x) \,\boldsymbol{t}\, 1 \leq \mu_B(x)$ and hence $\mu_A(x) \leq \mu_B(x)$, and for infinite universes $\mathcal{X}$ and left continuous $\boldsymbol{t}$ one has the same by (6), which means that in those cases from $[\![A \mathrel{\underline{\underline{\subset}}}_{\boldsymbol{t}} B]\!] = 1$ one gets $A \subseteq B$.

In the same manner the fuzzified identity can be defined as

$$A \equiv_{\boldsymbol{t}} B =_{def} A \mathrel{\underline{\underline{\subset}}}_{\boldsymbol{t}} B \wedge_{\boldsymbol{t}} B \mathrel{\underline{\underline{\subset}}}_{\boldsymbol{t}} A. \tag{10}$$

Both these fuzzified relations (8) and (10) in those formulas are defined for fuzzy sets in essentially the same way the corresponding notions can be defined in classical set theory for crisp sets. Only the language now has to be read in the sense of many-valued logic; cf. [4].

This generalized identity is a kind of fuzzy indistinguishability relation for fuzzy sets. Their negation, hence, should be something like a "measure of difference" for fuzzy sets, i.e. a kind of distance.

Finally we need the notion of a metric in $\mathbb{F}(\mathcal{X})$. A twoplace function ϱ from $\mathbb{F}(\mathcal{X})$ into the nonnegative reals $\mathbb{R}^+$ is a *metric* iff for all $A, B, C \in \mathbb{F}(\mathcal{X})$ the following conditions hold true:

(M1)	$\varrho(A, B) = 0$ iff $A = B$,	(identity property)
(M2)	$\varrho(A, B) = \varrho(B, A)$,	(symmetry)
(M3)	$\varrho(A, C) + \varrho(C, B) \geq \varrho(A, B)$.	(triangle inequality)

Sometimes the identity condition (M1) is weakened to the condition

(M1^p) $\varrho(A, A) = 0$.

By a *pseudo-metric* ϱ then a function is meant that fulfills conditions (M1^p), (M2), (M3).

2 t-norms and corresponding distances

Given a left continuous t-norm $\boldsymbol{t}$, a binary "distinguishability" function $\varrho_{\boldsymbol{t}}$ is defined on $I\!F(\mathcal{X})$ by

$$\varrho_{\boldsymbol{t}}(A,B) =_{def} 1 - [\![A \equiv_{\boldsymbol{t}} B]\!], \tag{11}$$

i.e. by always putting $\varrho_{\boldsymbol{t}}(A,B) = [\![\neg(A \equiv_{\boldsymbol{t}} B)]\!]$.

The problem of the present paper is to find a necessary and sufficient condition for $\boldsymbol{t}$ to yield via (11) a metric with properties (M1),..., (M3) as distinguishability function $\varrho_{\boldsymbol{t}}$.

The following theorem is the main result.

Theorem 1 *Suppose $\boldsymbol{t}$ is a left continuous t-norm. Then the function $\varrho_{\boldsymbol{t}}$ of (11) is a metric in $I\!F(\mathcal{X})$ iff $\boldsymbol{t} \geqq \boldsymbol{t}_{Łuk}$, i.e. iff for all $u, v \in [0,1]$:*

$$\max\{0, u+v-1\} \leq u\,\boldsymbol{t}\,v.$$

A close look at the way that theorem can be proven shows that it is enough to have definition (5) for a given t-norm $\boldsymbol{t}$, cf. [5].

Formally, but, this definition does not need the assumption of the left continuity of $\boldsymbol{t}$. Thus, perhaps this asumption is not necessary or can be weakened.

Indeed, the arguments to establish that $\varrho_{\boldsymbol{t}}$ has properties (M2) and (M3) do not use the left continuity of $\boldsymbol{t}$, cf. [2]. Thus, the crucial point is the proof of (M1). And for that case the following lemma holds true.

Lemma 1 *For $\varrho_{\boldsymbol{t}}(A,B) =_{def} 1 - [\![A \equiv_{\boldsymbol{t}} B]\!]$ it holds true*

$$\varrho_{\boldsymbol{t}}(A,B) = 0 \;\Leftrightarrow\; A = B \quad \text{for all } A, B \in I\!F(\mathcal{X})$$

iff each function $\boldsymbol{t}_u =_{def} \lambda v(u\,\boldsymbol{t}\,v)$, $u \in [0,1]$, is (left) continuous at $v = 1$.

This lemma therefore shows that the continuity assumption in the theorem can be weakened.

Furthermore, by the reflexivity of $\equiv_{\boldsymbol{t}}$ it is obvious that condition (M1^p), which instead of (M1) is used to define pseudo-metrics, always holds true for $\varrho_{\boldsymbol{t}}$.

Thus the following results hold true too.

Corollary 1 *ϱ_t is a metric iff $t \geqq t_{Luk}$ and each one of the functions $t_u = \lambda v(u\, t\, v)$ is left continuous at the point $v = 1$.*

Corollary 2 *ϱ_t is a pseudo-metric iff $t \geqq t_{Luk}$.*

Unfortunately, yet, it is not clear if these corollaries are as interesting as the theorem. The crucial point here is that in case of a t-norm t which is not left continuous, for φ_t the property

$$u\, t\, (u\, \varphi_t\, v) \leq v \quad \text{for all } u, v \in [0, 1] \tag{12}$$

fails, cf. [3]. But (12), one of the main properties which in [6] were used to define Φ-operators, means that the generalized conjunction $\wedge_t$ and the generalized implication $\rightarrow_t$ together fulfill a kind of generalized modus ponens: the truth degree of a well-formed formula H_2 always is not smaller as the truth degree of the $\wedge_t$-conjunction of formulas H_1 and $H_1 \rightarrow_t H_2$.

So we have the situation that the greater generality of the corollaries with respect to the theorem rests on the assumption that the modus ponens condition (12) is not an essential one for the context, the corollaries are used in.

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