COMPARATIVE STUDIES ON FUZZY T-NORM OPERATORS

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Abstract--To facilitate management and control, in this study, we analyze and compare nine of the most widely used fuzzy t-norm operators from six different aspects. Sensitivity analysis is carried out on the basic criterion of axiometic support. It is concluded that both Min and Yager's operators are comparatively superior to the others.

Keywords: t-norm operators, multiple criteria, sensitivity analysis

1. MOTIVATION AND PURPOSES

Define a fuzzy set as follows

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X \}, \quad 0 \leq \mu_{\tilde{A}}(x) \leq 1$$

then $\mu_{\tilde{A}}(x)$ is the membership of element x in the universe X. Because a fuzzy set is characterized by its membership function, therefore, when two fuzzy sets are aggregated, its properties are also characterized by the aggregated memberships. This aggregation normally is fullfilled by an operator. Thus, the results of aggregation are affected by the selected operator.

Thole et al. [5] has compared Min and Product operators with a special case and pointed out the importance of comparative studies. Wang [6] also applied quasi-Newton method to analyze Min and Hamacher's operators numerically. Because different operators possess different properties, it is essential to perform a systematic analysis so that their differences from both axiometic support and the application strength can be recognized. This is our aim of study.

2. METHODOLOGY

Because t-norm operators [2] listed in Table 1 are the most widely used operators in Fuzzy Set Theory, therefore, we adopt them as our objects of analysis. As regards the criteria of comparisom, Zimmermann [10] has proposed eight of them. However, "adaptability" is analogous to "aggregation behavior";"compensation" can be incorporated into the analysis of "range of compensation", six criteria listed in Table 2 are thus considered in this study. In addition, the levels of"imperical fit" for different operators can be case by case, therefore, we shall first consider the first five criteria. Then, Thole et al's case will be applied further to those comparatively better operators.

Before proceed the procedure of comparison, analysis of each operator with respect to each criterion is carried out. Then, they will be ranking ordered accordingly. Finally, an overall evaluation with five of these multiple criteria will be performed with equal weights. The first three of the ranked operators will be further evaluated with Thole's case. Sensitivity analysis is carried out especially on the criterion of "axiometic strength" and conclusions are drawn.

3. ANALYSIS AND COMPARISON

In this section, these operators are analyzed with respect to each criterion. The detailed proofs can be refered to Wang [7] and are omitted here. However,the methods of analyses and the measures of the levels of satisfaction will be described.

3.1 Axiometic Strength

An operator should satisfy some basic properties in operation so that it will have less limited in applications. Since Fuzzy Set Theory is induced and extended from the Crisp Set Theory and Boolean Algebra, it is natural to consider whether this extension still preserves the properties of a crisp set. Table 3 is listed the common properties of a crisp set where B1-B4 are the basic properties that are satisfied by all t-norm operators. the results of analysis are shown in Table 4 where the mark "X" represents the possession of that property by the corresponding operator.

Based on the criterion that when everythingelse being equal, an operator is better, if the more the axioms are which it satisfies, the last row of Table 4 shows the orders of comparison in which the Bounded and Yager's operators are comparatively superior.

3.2 Adaptability

It is apparent that there is no operator that can be applied every situation. However, if an operator that contains any parameter, it maybe more adaptable to the specific context. So, if an operator has no parameter, the relative degree of adaptability is zero. If one equals the other with certain values of the parameter, this operator is said to 'contain' the other one completely and the score of the degreeis added one. Otherwise, half unit of the scores will be added to each of the relevant operators to represent that they are equal when both of them are assigned a specific value to their parameters respectively.

Table 1. T-Norm Operators

Operators	Conjunctions	Disjunctions
1 Drastic operator	$T_w = \begin{cases} x & \text{if } y=1 \\ y & \text{if } x=1 \\ 0 & \text{otherwise} \end{cases}$	$T_w^* = \begin{cases} x & \text{if } y=0 \\ y & \text{if } x=0 \\ 1 & \text{otherwise} \end{cases}$
2 Bounded operator	$\max(0, x+y-1)$	$\min(1, x+y)$
3 Product operator	xy	$x+y-xy$
4 Yager operator [1] [8] ($p>0$)	$1-\min(1,[(1-x)^p+(1-y)^p]^{1/p})$	$\min(1,[x^p+y^p]^{1/p})$
5 Schweizer1 operator [4] ($p>0$)	$(x^{-p}+y^{-p}-1)^{-1/p}$	$1-((1-x)^{-p}+(1-y)^{-p}-1)^{-1/p}$
6 Schweizer2 operator [4] ($p<0$)	$[\max(0,x^{-p}+y^{-p}-1)]^{-1/p}$	$1-[\max(0,(1-x)^{-p}+(1-y)^{-p}-1)]^{-1/p}$
7 Hamacher operator [9] ($r\geq 0$)	$xy/(r+(1-r)(x+y-xy))$	$x+y-xy-(1-r)xy/(r+(1-r)(1-xy))$
8 Dubois operator [2] ($0\leq r\leq 1$)	$xy/\max(x,y,r)$	$x+y-xy-\min(1-r,x,y)/\max(r,1-x,1-y)$
9 Min operator	$\min(x,y)$	$\max(x,y)$

Table 2. Six Considered Criteria

1. Axiomatic Strength
2. Adaptability
3. Numerical Efficiency
4. Compensation and Its Range
5. Required Scale Level of Membership Function
6. Empirical Fit

Table 3. The Considered Axioms

axiom	function
B1 Commutativity	$i(x,y)=i(y,x)$
B2 Associativity	$i(i(x,y),z)=i(x,i(y,z))$
B3 Boundary Condition	$i(1,1)=1$, $i(1,0)=i(0,1)=i(0,0)=0$
B4 Monotonicity	If $x\leq x', y\leq y'$, then $i(x,y)\leq i(x',y')$
A1 Distributivity	$i(x,i'(y,z))=i'(i(x,y),i(x,z))$
A2 Idempotence	$i(x,x)=x$
A3 Identity	$i(x,1)=x$
A4 Law of Contradiction	$i(x,1-x)=0$
A5 Continuity	$\lim_{\Delta x\to 0, \Delta y\to 0} i(x+\Delta x, y+\Delta y)=i(x,y)$
A6 Additive Property	$i(x_1+x_2,y) = i(x_1,y) + i(x_2,y)$ $i(x,y_1+y_2) = i(x,y_1) + i(x,y_2)$
A7 Conservation	$\forall$ $k\in[0,1]$, so that $x-k$ & $y+k$ $\in[0,1]$, then $i(x,y)=i(x-k,y+k)$

Table 4. Analysis on the Axiomatic Strength

operator / axiom	1	2	3	4	5	6	7	8	9
B1	X	X	X	X	X	X	X	X	X
B2	X	X	X	X	X	X	X	X	X
B3	X	X	X	X	X	X	X	X	X
B4	X	X	X	X	X	X	X	X	X
A1									X
A2									X
A3	X	X		X	X	X	X	X	X
A4	X	X	X						
A5		X	X	X	X	X	X	X	X
A6			X						
A7		X							
order	3	1	2	3	3	3	3	3	1

For instance, the degree of adaptability for Min operator is zero, whereas that for Yarger's operator is three because it contains Drastic (Tw), Bounded and Min operators; and because Schweizer 1 with p=1 equals Hamacher's with r=0, so both get half unit of the scores.

Apart from those with no parameters that are all ranked at the fourth order, Table 5 shows the results of analysis and the ranked orders of comparison. It shows that the Yager's and Schweizer 2 operators are the leading ones on this aspect.

3.3 Numerical Efficiency

This criterion is focused on the computation effor an operator should be required.It is especially important when a large-scale problem is faced. The analysis is based on the measure of complexity in Data Structure where one 'unit-time' is defined by an operation one can accomplish with one step. Therefore, operators of +,-,x,%,> and < need one unit-time for each to accomplish the operation. As regardsthe power n of any number y, we can transform it into exp(n x (ln y)) that has "exp", "x" and "ln" three unit-time. If n elements are aggregated by Max or Min operator, in the worest case, it will require n-1 unit-time.

Then, based on the criterion that the less the unit-time an operator requires the better is the operator, Table 6 represents the results of analysis with the orders of comparison.

It can be noted that the operators of Product and Min are absolutely superior to the others. Those with parameters, only Dubois' operator is comparatively better.

Given the degree of membership to the aggregated fuzzy set by

$$\mu_{Agg}(x_k) = f(\mu_{\tilde{A}}(x_k),\ \mu_{\tilde{B}}(x_k)) = k, \qquad [10]$$

then f is compensatory if $\mu_{Agg}(x_k)=k$ for any $\mu_{\tilde{A}}$ & $\mu_{\tilde{B}}$. The larger the range of k,the more are the degree of compensation between the sets of $\tilde{A}$ and $\tilde{B}$, and the better is the operator.

The detailed analysis can be referred to Wang[7] and Table 7 is shown the results and their ranking orders.

It can be noticed that apart from the Min operator that is not compensatory, Tw operator has limited compensation when k=0; and Dubois' operator where the degree of compensation is directly related to the values of parameter r and the membership functions, the others are nearly equivalent in degrees.

3.5 Required Scale Level of Membership Functions

The criterion is emphasised on the easiness of obtainning the required information when we adopt some operator to aggregate two membership functions.

Normally the scale is classifed into five levels. According to the easiness of obtainning information, they are norminal, ordinal, difference, ratio and absolute [3]. Generally, the difficulty in obtainning information is compensated by the accuracy of information. However, from the viewpoint of information gathering, an operator that requires the lower level of scale is better.

Table 8 provides the results and the orders of comparison where Tw operator requires the lowest level of scale and thus, it is the best on this aspect. But it is noticed that the aggregated values with Tw are either 1 or 0. The applicability might be questionable.

Table 5. Analysis on the Adapability

operator	containness	degree	order
Yager	p → 0 : Tw p = 1 : max(0,x+y-1) p → ∞ : min(x,y)	3	1
Schweizer 1	p → 0 : xy p = 1 : xy/(x+y-xy) p → ∞ : min(x,y)	2.5	2
Schweizer 2	p → 0 : xy p = -1 : max(0,x+y-1) p →-∞ : Tw	3	1
Hamacher	r = 0 : xy/(x+y-xy) r = 1 : xy r → ∞ : Tw	2.5	2
Dubois	r = 0 : min(x,y) r = 1 : xy	2	3

Table 6. Analysis on Numerical Efficiency

operator	required steps	unit-time	order
Drastic operator	= (twice)	2	2
Bounded operator	+,-,max(2)	3	3
Product operator	x	1	1
Yager operator (p>0)	-,power,+,-,power,/ power,min(2),-	15	6
Schweizer1 operator (p>0)	-,power,+,-,power,- /,-,power	15	6
Schweizer2 operator (p<0)	-,power,+,-,power,- /,-,power,max(2)	16	7
Hamacher operator (r≧0)	x,/,+,-,x,+,-,x	8	5
Dubois operator (0≦r≦1)	x,/,max(3)	4	4
Min operator	min(2)	1	1

Table 7. Analysis on the Compensation with Orders

operator	range	order
Tw	$k = 0$	4
max(0,x+y−1)	$k \in [0,1)$	1
xy	$k \in (0,1)$	2
Yager	$k \in [0,1)$	1
Schweizer 1	$k \in (0,1)$	2
Schweizer 2	$k \in [0,1)$	1
Hamacher	$k \in [0,1)$	1
Dubois	$k \in (0,r)$	3
min(x,y)	$k \in \phi$	5

Table 8. Analysis on the Required Scale Level

operator \ scale	1	2	3	4	5	6	7	8	9
Nominal	△			$p \to 0$		$p \to -\infty$	$r \to \infty$		
Ordinal				$p \to \infty$	$p \to \infty$			$r=0$	△
Difference		△		$p=1$		$p=-1$			
Ratio			△		$p \to 0$	$p \to 0$	$r=1$	$r=1$	
Absolute				else	else	else	else	else	
order	1	5	6	2	4	2	2	4	3

4. MULTICRITERIA COMPARISON AND SENSITIVITY ANALYSIS

From the analysis above, it is noticed that in some aspects, the performance of some operators is better than that of others. No one is completely superior to the others. Therefore, they are nondominated alternatives. In order to compare them with all of critéria simultaneously, we proceed an overall evaluation by integrating these criteria with equal weights to avoid any bias. Table 9 is summerized the results of comparison with both single and multiple criteria. We notice that in the overall evaluation, Min, Bounded and Yager's operators lead the orders of comparison. However, when we adopt Thole et al's case data and carry out further analysis, we discover that the performance of the Bounded operator in this case is very bad (see Fig. 1). This fact tells us that besides the objective evaluation, the case studies for testing the criterion of "empirical fit" plays an considerable role in selecting an appropriate operator.

In addition, we have carried out sensitivity analysis on the criterion of the "axiom strength". This is because that alghough Fuzzy Set Theory is a kind of extension from Crisp Set Theory, it has its own properties. Therefore, the axiom like "the Law of Contradiction" should be reconsidered for the validity of a fuzzy set with vague boundaries. Then, the first row of ranking orders Table 10 represents the results when this axiom is dropped.

Furthermore,the property of "Conservation" is a special case of the criterion of "compensation ranges", in order to avoid doubly counted, this axiom has been further dropped. The second row of Table 10 shows this result.

It can be noticed that, in both cases the ranked orders of the first three operators are Min, Yager's and Schweizer 2.

5. SUMMARY AND CONCLUSIONS

In summary, in this study we compare the most widely used fuzzy t-norm operators with detailed analysis and proofs. Based on the six criteria,the part of analysis is carried out individually and the comparison is performed. Then, a global evaluation is done by first, aggregating five of these criteria with equal weights. This leads to the conclusion that the Min, Bounded and Yager's operators are the first three in ranking orders. Then, Thole et al's data are adopted to evaluate the "empirical fit" on these three operators.It is found that the performance of using Bounded It is found that the performance of using Bounded operator is rather bad. Therefore,one should be very concious in applications. Finally, sensitivity analyses on two axioms of the "Law of Contradiction" and "the conservation" are performed.

It is concluded that in overall, Min operator is the best both in theoretical support and in applications. For those parametric parameters, Yager's operator is compariatively superior.

Table 9: Ranked Orders with Single and Multiple Criteria

operator criterion	1	2	3	4	5	6	7	8	9
Axiomatic Strength	3	1	2	3	3	3	3	3	1
Adaptability	4	4	4	1	2	1	2	3	4
Numerical Efficiency	2	3	1	6	6	7	5	4	1
Compensation & Its Range	4	1	2	1	2	1	1	3	5
Required Scale Level	1	5	6	2	4	2	2	4	3
order	7	2	4	3	8	5	6	9	1

Table 10. Sensitivity Analyses

operator	1	2	3	4	5	6	7	8	9
order 1	5	2	4	3	3	3	3	3	1
order 2	7	6	5	2	8	3	4	9	1

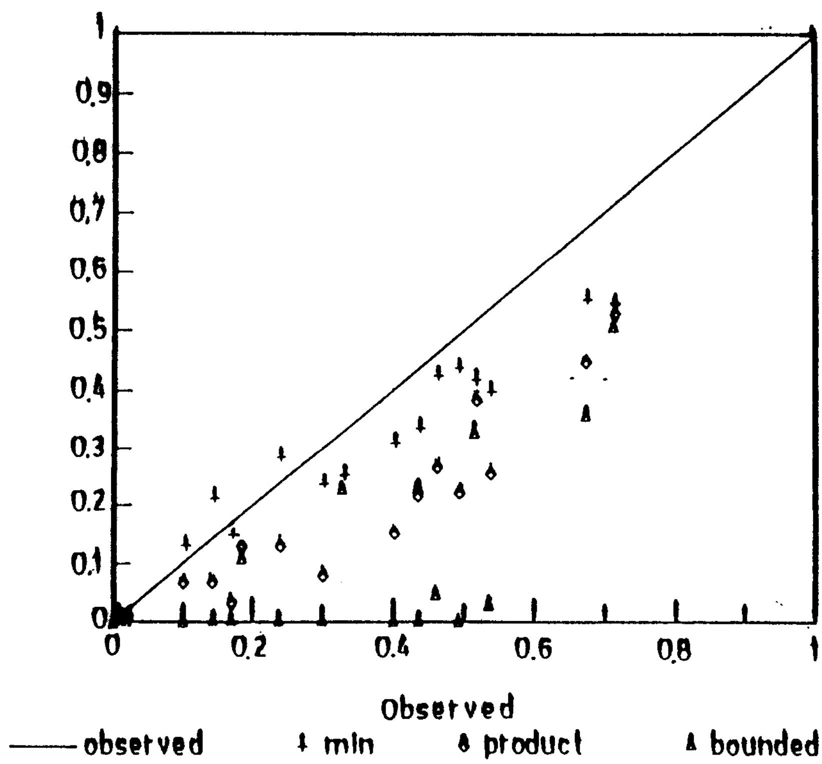

Fig. 1 Comparision of Three Operators with Empirical Case

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