APPROXIMATE REASONING BASED ON GENERALISED DISJUNCTIVE SYLLOGISM

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Abstract

After Zadeh introduced the concept of approximate reasoning (1) based on Generalised modus-ponens several authors/researchers used that to describe many different models. The similarity among them is that they are defined for well-defined propositions only. This paper proposes a generalisation of the widely used rule of inference in binary-valued logic, the law of disjunctive syllogism in order to tackle, in future, the undefined propositions effectively. A general purpose approximate reasoning technique based on generalised disjunctive syllogism is presented.

Key-words - Approximate reasoning, Generalised disjunctive syllogism.

1. Introduction

In the year 1965 J.A. Robinson (5) made a major breakthrough by introducing the complete resolution principle in case of two-valued logic. In this year again another major breakthrough was made by L.A. Zadeh who, for the first time, introduced the concept of fuzzy sets, a tool for handling soft-natured decision making problems effectively and efficiently. Since then many authors including Zadeh, Mamdani, Bandler, Baldwin, Yager, Mizumoto, Tsukatmoto, Kiszka (2)(4)(7) have been discussed fuzzy reasoning and their applications successfully. Again authors like R.C.T. Lee (6), M. Mukaidono (7) defined fuzzy resolution principle. Our idea is to present a generalised resolution principle that handles the inexact situation effectively and is applicable for both well-defined and undefined propositions. For that in this paper we present a technique of approximate reasoning based on the law of disjunctive syllogism.

In two-valued logic the law of disjunctive syllogism can be stated as -

'Given a disjunction and the negation of one of the disjuncts, the other may be inferred'. Symbolically,

$$\begin{array}{rl} \text{prem 1 :} & p \vee q \\ \text{prem 2 :} & \underline{\sim p} \\ \text{Concl :} & q \end{array}$$

In using this rule the user must be sure that the disjunct as appeared in the second premise is exactly the negetion of that as appeared in the first one. Let's remove this restriction (exactness) and generalize this concept in the case where all or some of the disjuncts are inexact propositions and hence the disjunct appeared in the second premise, in general, not exactly the negetion of that as appeared in the first premise.

As the first premise is a restriction of the disjuncts and the user may have some knowledge about any of the disjuncts it is always possible to infer the induced information regarding the other disjunct. The technique discribed is such that the obvious demand that the status of the inferred disjunct should exactly be the same as that occurred in the first restriction whenever the other disjunct as appeared in the second premise is exactly the negetion of that appeared in first premise is met. Thus, symbolically, we have

$$\begin{array}{rl} \text{prem 1 :} & p \vee q \\ \text{prem 2 :} & \underline{p'} \\ \text{Concl :} & q' \end{array}$$

where q' is exactly q whenever p' is identical with $\sim p$. This idea and its generalisation (the number of premises are more) provides an efficient technique for the establishment of the desired fuzzy resolution principle.

2.0 Mathematical Formulation of the Problem

In the first model we consider two typical premises expressed as

$$\begin{array}{l} p: \ X \text{ is } A \quad \text{or} \quad Y \text{ is } B \\ q: \ X \text{ is } A' \\ \hline \text{Conclusion: } Y \text{ is } B' \end{array}$$

where X, Y are two linguistic variables that define objects and A, B, A', B' are inexact concepts which are approximated by fuzzy sets over U, V, U, V respectively. U, V are the universe of discourse of X and Y respectively. Let p, q are translated into the possibility assignment equations

$$p \dashrightarrow \Pi_{(X,Y)} = R$$
$$\text{and } q \dashrightarrow \Pi_X = A'$$

where R is a fuzzy relation i.e. a fuzzy subset of the cartesian product UXV such that

$$\mu_R(u,v) = \min\left\{1 - \mu_A(u), \mu_B(v)\right\}.$$

The particularization of R by A' will be given by

$$\Pi_{(X,Y)}\left[\Pi_X = A'\right] = R \cap \bar{A}'$$

where $\bar{A}'$ is the cylindrical extension of A' i.e. $\bar{A}' = A' \times V$. Hence projection of $R \cap \bar{A}'$ on Y gives after retranslation the required inference

$$r \leftarrow \pi_Y = \pi_X \circ \pi_{(X,Y)}$$
$$= \mathrm{Proj}_Y\, \pi_{(X,Y)} \left[\pi_X = A'\right]$$

where $\quad \mu_{B'}(v) = \sup_u \left\{\mu_R(u,v) \wedge \mu_{A'}(u)\right\}.$

Now since

$$\pi_{(X,Y)} \left[\pi_X = A^c\right] = R \cap \bar{A}^c$$ (A^c denote the complemented fuzzy set of A)

we have, $\pi_Y = \mathrm{Proj}_Y\, \pi_{(X,Y)} \left[\pi_X = A^c\right]$

where $\mu_{B'}(v) = \sup_u \left\{\mu_R(u,v) \wedge \mu_{A^c}(u)\right\}$

$$= \sup_u \left\{\min\,(1-\mu_A(u), \mu_B(v)) \wedge \mu_{A^c}(u)\right\}$$
$$= \sup_u \left\{\min\,(1-\mu_A(u), \mu_B(v)) \wedge (1-\mu_A(u))\right\}$$
$$= \sup_u \left\{\min\,(1-\mu_A(u), \mu_B(v))\right\}$$
$$= \mu_B(v) \quad \text{iff} \quad \mu_B(v) \leq \sup\,(1-\mu_A(u)) \;.$$

This can be achieved iff we choose the fuzzy sets A and B in such a way that the membership values attains at least once both the bounds 0 and 1 at some points within their respective domains.

In case these bounds are not attained even then we will not lose any essential informations in the ultimate inference. Only the inferred results will be a very close approximation of the desired one. This phenomenon is also true in case of generalised modus ponens (1)(4).

In the second model there are two typical promises expressed as

$$\begin{array}{l} p : X_1 \text{ is } A_1 \text{ or } X_2 \text{ is } A_2 \text{ or ..... or } X_n \text{ is } A_n \\ q : X_1 \text{ is } A_1' \\ \hline \end{array}$$

and the conclusion : X_2 is A_2' or X_3 is A_3' or ... or X_n or A_n'

where $X_i(i = 1,2,\ldots,n)$ are n-variables (which specifies some objects) that range over finite sets or variables that are approximated by such sets. $A_i(i = 1,2,\ldots,n)$ and $A_i'(i = 1,2,\ldots,n)$ are inexact concepts that are approximated by fuzzy sets over $U_i(i = 1,2,\ldots,n)$. Let thembe

$$A_i = \sum_{j=1}^{j_i} \mu_{A_i}(u_i^j)/u_i^j \subset U_i = \sum_{j=1}^{j_i} u_i^j ; \quad i = 1,2,\ldots,n$$

$$A_i' = \sum_{j=1}^{j_i} \mu_{A_i'}(u_i^j)/u_i^j \subset U_i ; \quad i = 1,2,\ldots,n.$$

The translation of logical relations between sentences appearing in the premise p into mathematical relation gives

$$p \longrightarrow \pi_{(X_1,X_2,\ldots,X_n)} = R \subset U_1 X U_2 X \ldots X U_n$$

where

$$\mu_R(u_1,u_2,\ldots,u_n) = \min\left\{1 - \mu_{A_1}(u_1), \mu_{A_2}(u_2),\ldots,\mu_{A_n}(u_n)\right\}$$

and translation of the second premise gives

$$q \longrightarrow \pi_X = A_1' \subset U_1 .$$

The particularization of R by A_1' induced by the propositions p and q can be obtained as

$$\pi_{(X_1,X_2,\ldots,X_n)}\left[\pi_{X_1} = A_1'\right] = R\cap\bar{A}_1'$$

where $\bar{A}_1'$ is the cylindrical extension of A_1' over $U_1 \times U_2 \times \cdots \times U_n$

$$\text{i.e. } \bar{A}_1' = A_1' \times U_2 \times U_3 \times \ldots \times U_n.$$

Projecting $R\cap\bar{A}_1'$ Or $U_2 \times U_3 \times \ldots \times U_n$ we obtain

$$\pi_{(X_2,X_3,\ldots,X_n)} = \text{Proj}_{U_2\times U_3\times\ldots\times U_n}\left[R\cap\bar{A}_1'\right]$$

$$= S \text{ (, say)}$$

such that

$$\mu_S(u_2,u_3,\ldots,u_n) = \sup_{u_1}\left\{\mu_{A_1'}(u_1)\wedge\mu_R(u_1,u_2,\ldots,u_n)\right\}$$

$$= \sup_{u_1}\left\{\mu_{A_1'}(u_1)\wedge\min\left(1-\mu_{A_1}(u_1),\mu_{A_2}(u_2),\ldots,\mu_{A_n}(u_n)\right)\right\}$$

$$= \sup_{u_1}\left\{\min\left(\mu_{A_1'}(u_1),1-\mu_{A_1}(u_1),\mu_{A_2}(u_2),\ldots,\mu_{A_n}(u_n)\right)\right\}.$$

Hence we have a relation matrix S from the composition of propositions p and q. To obtain a more meaningful inference one can project S over V_i $(i=2,3,\ldots,n)$ one by one and obtain

$$\pi_i \leftarrow X_i \text{ is } A_i' \,;\quad i = 2,3,\ldots,n$$

where $A_i' = \text{Proj}_{U_i} S = \text{Proj}_{U_i}\pi_{(X_2,X_3,\ldots,X_n)}$

such that

$$\mu_{A_i'}(u) = \sup_{u_2,u_3,\ldots,u_{i-1},u_{i+1},\ldots,u_n}\left\{\mu_S\,(u_2,u_3,\ldots,u_{i-1},u\,,u_{i+1},\ldots u_n)\right\}.$$

Hence, after retranslation, the inference becomes

$$X_2 \text{ is } A_2' \text{ or } X_3 \text{ is } A_3' \text{ or } \ldots \text{ or } X_n \text{ is } A_n'.$$

In this case also it can be shown exactly in a similar manner that if A_1' is the complemented fuzzy set of A_1 then $A_i' = A_i$ for all i=2,3,...,n.

The third model concludes

$$X_2 \text{ is } A_2' \text{ or } X_3 \text{ is } A_3' \text{ or } \ldots \text{ or } X_m \text{ is } A_m' \text{ or } X_{m+1} \text{ is } A_{m+1}' \text{ or }$$

$$X_{m+1} \text{ is } A_{m+1}' \text{ or } \ldots \text{ or } X_n \text{ is } A_n'$$

from

prem 1 : X_1 is A_1 or X_2 is A_2 or ... or X_m is A_m

and prem 2 : X_1 is A_1' or X_{m+1} is A_{m+1} or ... or X_n is A_n .

It can be easily verified that the first two models are special cases of this (third) model. In this case we have

$$p \longrightarrow \pi_{(X_1,X_2,\ldots,X_m)} = R \subset U_1 \times U_2 \times \ldots \times U_m$$

where $\mu_R(u_1,u_2,\ldots,u_m) = \min\left\{1 - \mu_{A_1}(u_1),\ \mu_{A_2}(u_2),\ldots,\mu_{A_m}(u_m)\right\}$

and $q \longrightarrow \pi_{(X_1,X_{m+1},X_{m+2},\ldots,X_n)} = S \subset U_1 \times U_{m+1} \times \ldots \times U_n$

where $\mu_S(u_1,u_{m+1},\ldots,u_n) = \min\left\{\mu_{A_1'}(u_1),\ \mu_{A_{m+1}}(u_{m+1}),\ldots,\mu_{A_n}(u_n)\right\}$.

The particularization of R by S denoted by $\overline{R} \cap \overline{S}$ will be given by

$$\pi_{(X_1,X_2,\ldots,X_n)} = (R \times U_{m+1} \times \ldots \times U_n) \cap (S \times U_2 \times U_3 \times \ldots \times U_m) .$$

Hence the required inference will be given by

$$\pi_{(X_2,X_3,\ldots,X_m,X_{m+1},\ldots,X_n)} = \text{Proj}_{U_2 \times U_3 \times \ldots \times U_n}(\overline{R} \cap \overline{S})$$

$$= T \quad (\text{, say})$$

where, as usual,

$$\mu_T(u_2,u_3,\ldots,u_n) = \sup_{u_1}\left\{\mu_S \wedge \mu_R\right\}$$
$$= \sup_{u_1}\left\{\mu_S(u_1,u_{m+1},\ldots u_n) \wedge \mu_R(u_1,u_2,\ldots,u_m)\right\}.$$

Then projecting T on U_i for $i = 2,3,\ldots,n$ we obtain, at once

$$\pi_i \leftarrow X_i \text{ is } A_i'$$

where $$A_i' = \mathrm{Proj}_{U_i} T\ ;\quad i = 2,3,\ldots,n\ .$$

In this case also it can be shown that if A_i' is exactly the fuzzy complement of A_1 then

$$A_i' = A_i \quad \text{for all } i = 2,3,\ldots,n$$

which is exactly the demand for disjunctive syllogism.

Let us now consider a fourth, the last in this paper, model where from premises

prem 1 : X_1 is A_1 or X_2 is $A_2,\ldots,X_m$ is A_m

and prem 2 : X_1 is A_1' or X_{S_1} is A_{S_1} or ... or X_{S_k} is A_{S_k} or X_{m+1} is A_{m+1} or ... or X_n is A_n

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one may conclude : X_2 is A_2 or ... or X_{m+1} is A_{m+1} or ... or X_n is A_n

where $\{s_1,s_2,\ldots,s_k\}$ is a subsequence of the sequence $\{2,3,\ldots,m\}$. In this case, as before,

$$p \longrightarrow \Pi_{(X_1,X_2,\ldots,X_m)} = R \subset U_1 \times U_2 \times \ldots \times U_m$$

and $$\mu_R(u_1,u_2,\ldots,u_m) = \min\left\{1 - \mu_{A_1}(u_1), \mu_{A_2}(u_2)\ldots,\mu_{A_m}(u_m)\right\}$$

$$q \longrightarrow \Pi_{(X_1, X_{S_1}, \ldots, X_{S_k}, X_{m+1}, \ldots, X_n)} = S \subset U_1 \times U_{S_1} \times U_{S_2} \ldots \times U_{S_k} \times U_{m+1} \times \ldots \times U_n$$

and $\mu_S(u_1, u_{S_1}, \ldots, u_{S_k}, u_{m+1}, \ldots, u_n)$

$$= \min\left\{\mu_{A_1'}(u_1), \mu_{A_{S_1}}(u_{S_1}), \ldots, \mu_{A_{S_k}}(u_{S_k}), \mu_{A_{m+1}}(u_{m+1}) \ldots, \mu_{A_n}(u_n)\right\} .$$

The required inference will be given by

$$\Pi_{(X_2, \ldots, X_n)} = \text{Proj}_{U_2 \times U_3 \times \ldots \times U_n}(\bar{R} \cap \bar{S}) = T \quad (\text{, say})$$

where $\mu_T(u_2, u_3, \ldots, u_n) = \sup_{u_1, u_{S_1}, \ldots, u_{S_k}} \left\{ \mu_R(u_1, u_2, \ldots, u_m) \wedge \mu_S(u_1, u_{S_1}, \ldots, u_{S_k}, u_{m+1}, \ldots, u_n) \right\}$

and for some meaningful inference the projection of T over U_i, $i = 2,3,\ldots,n$ gives

$$r_i \leftarrow X_i \text{ is } A_i' \; ; \quad i = 2,3,\ldots,n$$

where

$$A_i' = \text{Proj}_{U_i} T \; ; \quad i = 2,3,\ldots,n$$

and ultimately a result

$$X_2 \text{ is } A_2' \text{ or } X_3 \text{ is } A_3' \text{ or } \ldots \text{ or } X_n \text{ is } A_n' .$$

It is again true that when A_1' is the fuzzy complement of

$$A_1, \; A_i' = A_i \; ; \quad i = 2,3,\ldots,n .$$

In all the above models, to make a meaningful resolvent, A_1' should be close enough to the complement of A_1. Similar phenomenon is also true for generalized modus ponens (9).

References

1. L.A. Zadeh, 'Theory of approximate reasoning' in Machine Intelligence-9 eds J.E. Hayes, Donald Michie and L.I. Mikulich, Ellis Horwood Limited (1970), pp.149-194.

2. E.H. Mamdani and B.R. Gaines, 'Fuzzy Reasoning and its application', Academic Press, N.Y., 1981.

3. H.J. Zimmerman, L.A. Zadeh and B.R. Gaines, 'Fuzzy sets and decision analysis', North-Holland, 1984.

4. M.M. Gupta, A. Kandel, W. Bandler and J.B. Kiszka, 'Approximate Reasoning in Expert System', North Holland, 1985.

5. J.A. Robinson, 'A Machine oriented logic based on the resolution principle', J. ACM, 12, 1, pp.23-41, 1965.

6. R.C.T. Lee, 'Fuzzy logic and the resolution principle', J. ACM, 19, 1, pp.109-119, 1972.

7. M. Mukaidono, 'Fuzzy inference of resolution style, in Fuzzy Set and possibility theory, eds R.R. Yager, Pergamon Press, N.Y. (1982), pp.224-231.

8. Z. Shem, L. Ding and M. Mukaidono, 'A theoretical framework of fuzzy prolog machine, Fuzzy Computer, eds. M.M. Gupta and T. Yama Kawa, North Holland, 1988.

9. D. Dubois and H. Prade, 'On distance between fuzzy points and their use for plausible reasoning', Proceeding of IEEE International Conf. on SMC, Vol.1, Bombay-Delhi, pp.300-303, 1984.