On Pointwise Depictions of Fuzzy Relations

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Abstract

In this paper, we give the pointwise depictions of a fuzzy relation, the composition of two fuzzy relations, a fuzzy equivalence relation, and introduce the definition of fuzzy compatible which differ from N. Kuroki's. Also we obtain the pointwise depictions of fuzzy left (right) compatible and fuzzy compatible.

1. Introduction

The concept of a fuzzy relation on a set was defined by Zadeh [1, 2] and several authors have considered it further (see for example, Rosenfeld [3], Bhattacharya and Mukherjee [4] and Kuroki [5] etc.). In the present paper, we consider fuzzy relations on a semigroup S (cf. [6]), the pointwise depictions of a fuzzy relation, a fuzzy equivalence relation and the composition of two fuzzy relations are given. We define fuzzy compatible, which differ from Kuroki's in [5], on a semigroup S. Also we obtain the pointwise depictions of fuzzy left (right) compatible and fuzzy compatible.

2. Preliminaries

We review briefly some definitions and results about fuzzy set on S.

A map f from S to [0, 1] is called a fuzzy set on S. For any fuzzy sets A, B, A_i on S, where $i \in T$ (indexing set):

$$A \subseteq B \text{ iff } A(x) \leq B(x) \qquad \forall x \in S;$$

$$(\bigcup_{i \in T} A_i)(x) = \sup_{i \in T} A_i(x) \qquad \forall x \in S.$$

The product for two fuzzy sets A, B on S occuring in [7] is defined by

$$(A\circ B)(x) = \begin{cases} \sup\limits_{x=yz} \min\{A(y),\ B(z)\} & \text{for } y,\ z\in S,\ x = yz, \\ 0 & \text{for any } y,\ z\in S,\ x \neq yz, \end{cases}$$

for all $x\in S$.

Pu and Liu gave the definition of a fuzzy point (cf. [8]), that is, a fuzzy set on S x_λ is called a fuzzy point iff

$$x_\lambda(y) = \begin{cases} \lambda & y = x, \\ 0 & y \neq x, \end{cases}$$

for all $y\in S$, where $\lambda\in(0,\ 1]$. For x_λ and y_μ, $x_\lambda \subseteq y_\mu$ iff $x = y$ and $\lambda \leq \mu$. For a fuzzy set A on S, $x_\lambda\in A$ iff $x_\lambda(y) \leq A(y)$ for all $y\in S$.

Proposition 2.1. Let x_λ, y_μ are two fuzzy points on S, then

$$x_\lambda \circ y_\mu = (xy)_{\min\{\lambda,\ \mu\}}.$$

Proof. It is easily (cf. [7]).

3. Pointwise Depictions of fuzzy relations and their composition

Definition 3.1. A map ρ from $S\times S$ to $[0,\ 1]$ is called a fuzzy relation on S.

Obviously a fuzzy relation ρ on S is also a fuzzy set on $S\times S$.

In the following, let δ_S denote the characteristic function of S. It is well known that δ_S is the greatest fuzzy set on S, and can be expressed as the union of all the fuzzy points on S. Hence we can write

$$\delta_S = \{x_\lambda :\ x\in S,\ \lambda\in(0,\ 1]\}.$$

Definition 3.2. Let a_λ, $b_\mu\in\delta_S$, we define $(a_\lambda,\ b_\mu)$ as follows:

$$(a_\lambda,\ b_\mu)(x,\ y) = \min\{a_\lambda(x),\ b_\mu(y)\}$$

for all $x,\ y\in S$.

Clearly that $(a_\lambda,\ b_\mu)$ is a fuzzy relation on S and the Cartesian product of a_λ and b_μ (cf. [4]).

Definition 3.3. Let ρ be a fuzzy relation on S. For a_λ, $b_\mu\in\delta_S$, we write $(a_\lambda,\ b_\mu)\in\rho$ iff $(a_\lambda,\ b_\mu)(x,\ y) \leq \rho(x,\ y)$ for all $x,\ y\in S$.

By the following proposition, we give a pointwise depiction of a fuzzy relation on S.

Proposition 3.4. If ρ is a fuzzy relation on S, then

$$\rho = \bigcup_{\substack{(a_\lambda,\ b_\mu)\in\rho \\ a_\lambda,\ b_\mu \in \sigma_S}} (a_\lambda,\ b_\mu).$$

Proof. By Definition 3.3, we only need to prove

$$\rho(x,\ y) \leq \Big[\bigcup_{(a_\lambda,\ b_\mu)\in\rho} (a_\lambda,\ b_\mu)\Big](x,\ y)$$

for all x, $y \in S$.

Let x, y be any elements of S. If $\rho(x,\ y) = 0$, then

$$\rho(x,\ y) = 0 \leq \Big[\bigcup_{(a_\lambda,\ b_\mu)\in\rho} (a_\lambda,\ b_\mu)\Big](x,\ y).$$

If $\rho(x,\ y) \neq 0$, let $t = \rho(x,\ y)$, then $(x_t,\ y_t) \in \rho$. So

$$\begin{aligned}\Big[\bigcup_{(a_\lambda,\ b_\mu)\in\rho} (a_\lambda,\ b_\mu)\Big](x,\ y) &= \sup_{(a_\lambda,\ b_\mu)\in\rho} \min\{a_\lambda(x),\ b_\mu(y)\} \\ &\geq \min\{x_t(x),\ y_t(y)\} \\ &= t.\end{aligned}$$

This completes the proof.

Definition 3.5. Let ρ be a fuzzy relation on S. We define $a_\lambda \rho\, b_\mu$ for $a_\lambda,\ b_\mu \in \sigma_S$ iff $(a_\lambda,\ b_\mu) \in \rho$.

Definition 3.6. Let ρ and σ be two fuzzy relations on S. The composition $\rho\circ\sigma$ of ρ and σ is defined by

$$(\rho\circ\sigma)(x,\ y) = \sup_{z\in S} \min\{\rho(x,\ z),\ \sigma(z,\ y)\}$$

for all x, $y \in S$.

Proposition 3.7. Let ρ and σ be two fuzzy relations on S. Then $a_\lambda \rho\circ\sigma\, b_\mu$ for a_λ, $b_\mu \in \sigma_S$ iff there exist some $z_t \in \sigma_S$ such that $a_\lambda \rho\, z_t$, $z_t \sigma\, b_\mu$, and

$$\sup_{\substack{z_t \\ a_\lambda\rho z_t,\ z_t\sigma b_\mu}} t = \min\{\lambda,\ \mu\}.$$

Proof. Let $a_\lambda,\ b_\mu \in \sigma_S$. If $a_\lambda \rho\circ\sigma\, b_\mu$, then $(a_\lambda,\ b_\mu) \in \rho\circ\sigma$. So

$$(\rho\circ\sigma)(a,\ b) \geq \min\{\lambda,\ \mu\},$$

that is

$$\sup_{z \in S} \min\{\rho(a, z), \sigma(z, b)\} \geqslant \min\{\lambda, \mu\}. \qquad \text{(i)}$$

Then for a given ε, where $0 < \varepsilon < \min\{\lambda, \mu\}$, there must exist a $z \in S$ such that

$$\min\{\rho(a, z), \sigma(z, b)\} \geqslant \min\{\lambda, \mu\} - \varepsilon.$$

Otherwise we have

$$\min\{\rho(a, z), \sigma(z, b)\} < \min\{\lambda, \mu\} - \varepsilon$$

for all $z \in S$. Thus

$$\sup_{z \in S} \min\{\rho(a, z), \sigma(z, b)\} \leq \min\{\lambda, \mu\} - \varepsilon,$$

hence

$$\sup_{z \in S} \min\{\rho(a, z), \sigma(z, b)\} < \min\{\lambda, \mu\}.$$

This contradicts (i). Therefore we can assume that for some $z \in S$

$$\min\{\rho(a, z), \sigma(z, b)\} \geqslant \min\{\lambda, \mu\} - \varepsilon.$$

That is

$$\rho(a, z) \geqslant \min\{\lambda, \mu\} - \varepsilon \text{ and } \sigma(z, b) \geqslant \min\{\lambda, \mu\} - \varepsilon .$$

Set $t = \min\{\lambda, \mu\} - \varepsilon$, then $z_t \in \sigma_S$ and $a_\lambda \rho z_t$, $z_t \sigma b_\mu$. Obviously from above we have

$$\sup_{\substack{z_t \\ a_\lambda \rho z_t,\ z_t \sigma b_\mu}} t = \sup_{0 < \varepsilon < \min\{\lambda, \mu\}} (\min\{\lambda, \mu\} - \varepsilon) = \min\{\lambda, \mu\}.$$

Conversely, if there exist some $z_t \in \sigma_S$ such that $a_\lambda \rho z_t$, $z_t \sigma b_\mu$, and

$$\sup_{\substack{z_t \\ a_\lambda \rho z_t,\ z_t \sigma b_\mu}} t = \min\{\lambda, \mu\},$$

then

$$\rho(a, z) \geqslant \min\{\lambda, t\},\ \sigma(z, b) \geqslant \min\{t, \mu\}.$$

So

$$\begin{aligned} \min\{\rho(a, z), \sigma(z, b)\} &\geqslant \min\{\min\{\lambda, t\}, \min\{t, \mu\}\} \\ &= \min\{\lambda, t, \mu\}. \end{aligned}$$

Then we have

$$\begin{aligned}\sup_{y\in S}\min\{\rho(a, y), \sigma(y, b)\} &\geqslant \sup_{\substack{z_t \\ a_\lambda\rho z_t,\ z_t\sigma b_\mu}}\min\{\rho(a, z), \sigma(z, b)\} \\ &\geqslant \sup_{\substack{z_t \\ a_\lambda\rho z_t,\ z_t\sigma b_\mu}}\min\{\lambda, t, \mu\} \\ &= \min\{\lambda, \sup_{\substack{z_t \\ a_\lambda\rho z_t,\ z_t\sigma b_\mu}} t, \mu\} \\ &= \min\{\lambda, \min\{\lambda, \mu\}, \mu\} \\ &= \min\{\lambda, \mu\}.\end{aligned}$$

That is $(\rho\circ\sigma)(a, b) \geqslant \min\{\lambda, \mu\}$ by Definition 3.6, hence $a_\lambda\,\rho\circ\sigma\, b_\mu$.

Definition 3.8. A fuzzy relation ρ on S is called fuzzy reflexive if $\rho(x, x) = 1$ for all $x \in S$, and is called a fuzzy symmetric if $\rho(x, y) = \rho(y, x)$ for all $x, y \in S$, and is called fuzzy transitive if $\rho \geq \rho\circ\rho$. A fuzzy relation ρ on S is called a fuzzy equivalence relation if it is fuzzy reflexive, fuzzy symmetric and fuzzy transitive.

Proposition 3.9. Let ρ be a fuzzy relation on S. Then

(1) ρ is fuzzy reflexive iff $(x_t, x_t)\in\rho$ for all $x_t \in \sigma_S$;

(2) ρ is fuzzy symmetric iff $(x_\lambda, y_\mu)\in\rho$ for $x_\lambda, y_\mu \in \sigma_S$ implies $(y_\mu, x_\lambda)\in\rho$;

(3) ρ is fuzzy transitive iff for $x_\lambda, z_t \in \sigma_S$, if there exist some $y_\mu \in \sigma_S$ such that $x_\lambda\rho y_\mu$, $y_\mu\rho z_t$, and

$$\sup_{\substack{y_\mu \\ x_\lambda\rho y_\mu,\ y_\mu\rho z_t}} \mu = \min\{\lambda, t\},$$

then $x_\lambda \rho z_t$.

Proof. (1) can be obtained immediately from Definition 3.8 and 3.3.

(2) Let ρ be fuzzy symmetric. If $(x_\lambda, y_\mu)\in\rho$ for $x_\lambda, y_\mu\in\sigma_S$, then

$$\rho(y, x) = \rho(x, y) \geqslant \min\{\lambda, \mu\} = \min\{\mu, \lambda\}$$

since ρ is a fuzzy symmetric. That is $(y_\mu, x_\lambda)\in\rho$. Now we assume if $(x_\lambda, y_\mu)\in\rho$ for $x_\lambda, y_\mu\in\sigma_S$ implies $(y_\mu, x_\lambda)\in\rho$. Then there must have $\rho(x, y) = \rho(y, x)$ for all $x, y \in S$. Otherwise, there exist $x^*, y^* \in S$ such that

$$\rho(x^*, y^*) \neq \rho(y^*, x^*).$$

Assume

$$\rho(x^*, y^*) < \rho(y^*, x^*) \qquad \text{(ii)}$$

without loss of generality. Let $t = \rho(y^*, x^*)$, then $(y^*_t, x^*_t) \in \rho$. So $(x^*_t, y^*_t) \in \rho$ by hypothesis, that is

$$\rho(x^*, y^*) \geqslant t = \rho(y^*, x^*).$$

This contradicts (ii). This follows that $\rho(x, y) = \rho(y, x)$ for all $x, y \in S$.

(3) Necessity. For $x_\lambda, z_t \in \sigma_S$, if there exist some $y_\mu \in \sigma_S$ such that $x_\lambda \rho y_\mu$, $y_\mu \rho z_t$, and

$$\sup_{\substack{y_\mu \\ x_\lambda \rho y_\mu,\ y_\mu \rho z_t}} \mu = \min\{\lambda, \mu\},$$

then for $x_\lambda, z_t \in \sigma_S$ we have $x_\lambda \rho\circ\rho z_t$ by Proposition 3.7. That is

$$\rho(x, z) \geqslant (\rho\circ\rho)(x, z) \geqslant \min\{\lambda, t\}$$

since ρ is fuzzy transitive. Hence $x_\lambda \rho z_t$.

Sufficiency. Let x, y be any elements of S. If $(\rho\circ\rho)(x, y) = 0$, then

$$(\rho\circ\rho)(x, y) = 0 \leq \rho(x, y).$$

If $(\rho\circ\rho)(x, y) \neq 0$, let $(\rho\circ\rho)(x, y) = t$, then $x_t \rho\circ\rho y_t$. By Proposition 3.7 there exist some $z_\lambda \in \sigma_S$ such that $x_t \rho z_\lambda$, $z_\lambda \rho y_t$, and

$$\sup_{\substack{z_\lambda \\ x_t \rho z_\lambda,\ z_\lambda \rho y_t}} \lambda = \min\{t, t\} = t.$$

Therefore by hypothesis we have $x_t \rho y_t$, that is

$$\rho(x, y) \geqslant t = (\rho\circ\rho)(x, y).$$

This shows $\rho(x, y) \geqslant (\rho\circ\rho)(x, y)$ for all $x, y \in S$. That is $\rho \supseteq \rho\circ\rho$.

4. Fuzzy compatible and their pointwise depictions

Definition 4.1. A relation R on S is called left compatible if $(a, b) \in R$ implies $(xa, xb) \in R$ for all a, b and x of S, and is called right compatible if $(a, b) \in R$ implies $(ax, bx) \in R$ for all a, b and x of S. It is called compatible if R is both left and right compatible (cf. [6]).

Definition 4.2. A fuzzy relation ρ on S is called fuzzy left compatible (cf. [5]) if $\rho(xa, xb) \geqslant \rho(a, b)$ for all a, b and x of S, and is called fuzzy right compatible (cf. [5]) if $\rho(ax, bx) \geqslant \rho(a, b)$ for all a, b and x of S. It is called fuzzy compatible if ρ is both fuzzy left and fuzzy right compatible.

Proposition 4.3. Let ρ be a fuzzy relation on S. Then the following conditions are equivalent:

(1) ρ is fuzzy left (right) compatible;

(2) For each $t \in [0, 1]$, if $R_\rho(t) = \{(a, b): (a, b) \in S \times S, \rho(a, b) \geqslant t\} \neq \phi$, then $R_\rho(t)$ is left (right) compatible.

Proof. First assume (1) holds. For each $t \in [0, 1]$, if $R_\rho(t) \neq \phi$, then if $(a, b) \in R_\rho(t)$, we have $\rho(a, b) \geqslant t$. Thus for any $x \in S$, $\rho(xa, xb) \geqslant \rho(a, b) \geqslant t$ since ρ is fuzzy left compatible. So $(xa, xb) \in R_\rho(t)$ for all x of S. This means (1) implies (2).

Conversely, suppose that ρ is not fuzzy left compatible, then there must have three elements x^*, a, $b \in S$ such that $\rho(x^*a, x^*b) < \rho(a, b)$. Let

$$\lambda_0 = \frac{1}{2}\{\rho(x^*a, x^*b) + \rho(a, b)\},$$

thus $0 \leq \rho(x^*a, x^*b) < \lambda_0 < \rho(a, b) \leq 1$. That is $(a, b) \in R_\rho(\lambda_0)$, further $R_\rho(\lambda_0) \neq \phi$ for $\lambda_0 \in (0, 1]$. By (2) holds, we have $R_\rho(\lambda_0)$ is left compatible, then $(x^*a, x^*b) \in R_\rho(\lambda_0)$ for $x^* \in S$. So that $\rho(x^*a, x^*b) \geqslant \lambda_0$. This is in contradiction with $\rho(x^*a, x^*b) < \lambda_0$. This examines that (2) implies (1).

In the similar way we can prove the right case.

From Proposition 4.3, Definition 4.1 and 4.2 we have:

Proposition 4.4. A fuzzy relation ρ on S is fuzzy compatible iff for each $t \in [0, 1]$, if $R_\rho(t) = \{(a, b): (a, b) \in S \times S, \rho(a, b) \geqslant t\} \neq \phi$, then $R_\rho(t)$ is a compatible relation on S.

Proposition 4.5. Let ρ be a fuzzy equivalence relation on S. Then the following conditions are equivalent:

(1) $\rho(ac, bd) \geqslant \min\{\rho(a, b), \rho(c, d)\}$ for all a, b, c, d of S;

(2) ρ is fuzzy compatible.

Proof. First assume (1) holds. Let a, b, c be any elements of S. Then $\rho(c, c) = 1$, so that

$$\rho(ca, cb) \geqslant \min\{\rho(c, c), \rho(a, b)\} = \rho(a, b),$$
$$\rho(ac, bc) \geqslant \min\{\rho(a, b), \rho(c, c)\} = \rho(a, b).$$

Therefore ρ is fuzzy compatible, and (1) implies (2).

Now assume (2) holds. Let a, b, c, d be any elements of S. Then

$$\begin{aligned}\rho(ac, bd) &\geqslant (\rho\circ\rho)(ac, bd) \\ &= \sup_{z \in S} \min\{\rho(ac, z), \rho(z, bd)\} \\ &\geqslant \min\{\rho(ac, bc), \rho(bc, bd)\} \\ &\geqslant \min\{\rho(a, b), \rho(c, d)\}\end{aligned}$$

since ρ is a fuzzy equivalence relation and fuzzy compatible on S. Thus (2) implies (1).

Remark. In Proposition 4.5, if ρ is not fuzzy equivalent but a fuzzy relation on S, then generally (1) can't implies (2). That is, the concept of fuzzy compatible in Definition 4.2 is not equivalent to the definition of fuzzy compatible in [5] (in [5], a fuzzy relation ρ on S is called fuzzy compatible if

$$\rho(ac, bd) \geqslant \min\{\rho(a, b), \rho(c, d)\}$$

for all a, b, c and d of S).

Example. Let $S = \{x, y\}$, the binary operation on S is defined as follows:

$\cdot$	x	y
x	x	y
y	y	y

Evidently S is a semigroup with respect to the binary operation as above. We define f satisfying:

$$f(x, x) = 0.5,\ f(x, y) = f(y, x) = 0.6,\ f(y, y) = 0.4.$$

We can test that f is a fuzzy relation on S, and

$$f(ac, bd) \geqslant \min\{f(a, b), f(c, d)\}$$

for all a, b, c, d of S. But

$$f(xy, xy) = f(y, y) = 0.4, \ f(x, x) = 0.5.$$

That is $f(xy, xy) \ngeqslant f(x, x)$. This means f is not fuzzy right compatible. Hence f is not fuzzy compatible in the sense of Definition 4.2.

In the following, we will give the pointwise depictions of fuzzy left (right) compatible and fuzzy compatible.

Proposition 4.6. Let ρ be a fuzzy relation on S. Then the following conditions are equivalent:

(1) ρ is fuzzy left (right) compatible;

(2) If $a_\lambda \rho b_\mu$ for $a_\lambda, b_\mu \in \delta_S$, then $d_t \circ a_\lambda \rho d_t \circ b_\mu$ $(a_\lambda \circ d_t \rho b_\mu \circ d_t)$ for every $d_t \in \delta_S$.

Proof. First let (1) hold. If $a_\lambda \rho b_\mu$ for $a_\lambda, b_\mu \in \delta_S$, then for every $d_t \in \delta_S$, $\rho(da, db) \geqslant \rho(a, b) \geqslant \min\{\lambda, \mu\} \geqslant \min\{\min\{t, \lambda\}, \min\{t, \mu\}\}$. That is

$$((da)_{\min\{t, \lambda\}}, (db)_{\min\{t, \mu\}}) \in \rho.$$

Hence $(d_t \circ a_\lambda, d_t \circ b_\mu) \in \rho$ by Proposition 2.1. This follows that $d_t \circ a_\lambda \rho d_t \circ b_\mu$ for every $d_t \in \delta_S$.

Now assume (2) holds. Let a, b be any elements of S. If $\rho(a, b) = 0$, then obviously $\rho(xa, xb) \geqslant 0 = \rho(a, b)$ for all $x \in S$; If $\rho(a, b) \neq 0$, let $t = \rho(a,b)$, then $a_t \rho b_t$. By hypothesis we have $x_t \circ a_t \rho x_t \circ b_t$ for all $x \in S$. That is $(xa)_t \rho (xb)_t$. Hence $\rho(xa, xb) \geqslant \min\{t, t\} = t = \rho(a, b)$ for all x, a, b of S. So ρ is fuzzy left compatible. That is (2) implies (1).

Similarly we can verify the right case.

From Proposition 4.6 and Definition 4.2, we have:

Proposition 4.7. Let ρ be a fuzzy relation on S. Then the following conditions are equivalent:

(1) ρ is fuzzy compatible;

(2) If $a_\lambda \rho b_\mu$ for $a_\lambda, b_\mu \in \delta_S$, then $d_t \circ a_\lambda \rho d_t \circ b_\mu$ and $a_\lambda \circ d_t \rho b_\mu \circ d_t$ for every $d_t \in \delta_S$.

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