Fubini Theorem of F-valued Integrals

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Abstract: In the paper, further investigations of set-valued integrals are carried out. After proving Fubini theorem of set-valued integrals, Fubini theorem of F-valued integrals is given. They are the extension of classical Fubini theorem.

Keywords: set-valued integral, F-valued integral, Fubini theorem.

1. Introduction

Recently, integrals of fuzzy-valued functions (Simply said to be F-valued integrals) were investigated by Prof. Dubois and Prade [3] , Puri and Ralescu [4] and kaleva [5] . The paper's purpose is continuing their works, to develop the double integrals, and the main result is Fubini theorem. Up to now, there are no attempts in the work of the authors mentioned above. Since a F-valued function is essentialy a family of set-valued functions. We utilize results of set-valued functions.

In section 2, as the preparation we prove Fubini theorem of integrals of set-valued functions (Simply said to be set-valued integrals). Also the set-valued function is only valued in coR^n (the family of all nonempty compact convex subsets of R^n) as the restriction.

In section 3, we first define the F-valued integrals, where the F-valued function is valued in $E^n = \{u: R^n \to [0,1] \mid [u]^\alpha = \{x \in R^n;\ u(x) \geq \alpha\} \in coR^n$ for all $\alpha \in (0,1]\}$. Then we obtain Fubini theorem.

Throughout the paper, let (X, Σ, μ) and (Y, Σ', μ') be two complete σ-finite measure space, and $(X \times Y, \Sigma \times \Sigma', \mu \times \mu')$ be their product measure space.

2. Fubini theorem of set-valued integrals

In this section, we briefly state some concepts and results related to set-valued integrals. Most of them can be found in [1]. At last, a new result ——Fubini theorem is proved.

Let symbol coR^n denote the family of all nonempty compact convex subsets of Euclidean R^n, $P(R^n)$ denote the power set of R^n. A set-valued function $F: X \to P(R^n) \setminus \{\phi\}$ is called Σ-measurable if $F^{-1}(B) = \{x \in X: F(x) \cap B \neq \phi\} \in \Sigma$ for every Borel set B of R^n; a Σ-measurable set-valued function F is called μ-integrably bounded if there exists an μ-integrable function $h: X \to R^+$, such that $\|v\| \leq h(x)$ for all v, x with $v \in F(x)$. Let $S(F, \mu) = \{f$ is μ-integrable, $f(x) \in F(x),\ x \in X\ \mu - a \cdot e\}$, define

$$\int_X F(x)d\mu = \{\int_X f d\mu,\ f \in S(F, \mu)\}$$

Lemma 2.1 [2] A set-valued function $F: X \to coR^n$ is Σ-measurable if and only if its support function $\sigma(v \mid F): X \to R$ is Σ-measurable for all $v \in R^n$, where

$$\sigma(v \mid F)(x) = \sup\ \{(q, v);\ q \in F(x)\}$$

Lemma 2.2 [2] If a set-valued function $F: X \to coR^n$ is μ-integrably bounded, then $\int_X F(x)d\mu \in coR^n$,

$$\sigma(v \mid \int_X F(x)d\mu) = \int_X \sigma(v \mid F)d\mu$$

for all $v \in R^n$

Remark, All the concepts and lemmas above are suitable for (Y, Σ', μ') and $(X\times Y, \Sigma\times\Sigma', \mu\times\mu')$.

Having the above preparation, we can prove Fubini theorem of set-valued integrals.

Theorem 2.1.(Fubini th.) If a set-valued function $F: X\times Y \to coR^n$ is $\mu\times\mu'$-integrably bounded, then

(1) the set-valued function $F(\cdot, y): x \to F(x, y)$ is μ-integrably bounded for $y \in Y$ μ'-a, e;

(2) the set-valued function $I(\cdot): y \to I(y) = \int_X F(x, y)\,d\mu$ is μ'-integrably bounded, and

$$\int_{X\times Y} F(x, y)d\mu\times\mu' = \int_Y I(y)d\mu'$$

Proof. Since $F(\cdot, \cdot)$ is $\Sigma\times\Sigma'$-measurable, by lemma 2, 1. we can see that $\sigma(v \mid F): X\times Y \to R$ is $\Sigma\times\Sigma'$-measurable for every $v \in R^n$. Therefore, by classical Fubini theorem, we know that $\sigma(v \mid F)(\cdot, y): X \to R$ is Σ-measurable for $y \in Y$ μ'-a.e. Using lemma 2.1 again, we obtain that $F(., y)$ is Σ-measurable for $y \in Y$ μ-a.e. Consequently, (1) is naturally held.

By lemma 2, 2 , we get that $I(y) = \int_X F(x, y)d\mu \in coR^n$, and

$$\sigma(v \mid I)(y) = \int_X (v \mid F)(., y)d\mu$$

for all $v \in R^n$, Furthermore, function $\sigma(v \mid I)$ is Σ'-measurable by classical Fubini theorem, then set-valued function $I(\cdot)$, $y \to I(y)$ is Σ'-measurable by lemma 2, 1. By the given condition that there exists a $\mu\times\mu'$-integrable function $\phi: X\times Y \to R$, s. t. $\|v\| \leq \phi(x, y)$ for all $v, (x, y)$ with $v \in F(x, y)$, consequently

$$\sup_{v \in I(y)} \|v\| \leq \sup \{ \int_X \|f(x, y)\| d\mu ; f(., y) \in S(F(., y), \mu) \}$$

$$\leq \int_X \phi(., y) d\mu$$

That is to say, $I(.)$ is μ'-integrably bounded. Since for every $v \in R^n$, holds equations

$$\begin{aligned} \sigma(v \mid \int_{X\times Y} F(x, y) d\mu \times \mu') &= \int_{X\times Y} \sigma(v \mid F) d\mu \times \mu' \\ &= \int_Y \int_X \sigma(v \mid F) d\mu d\mu' \\ &= \int_Y \sigma(v \mid I) d\mu' \\ &= \sigma(v \mid \int_Y I(y) d\mu') \end{aligned}$$

By the convexity of $\int_{X\times Y} F(x, y)\, d\mu \times \mu'$ and $\int_Y I(y) d\mu'$, (ii) is proved. (Q.E.D)

3. Fubini theorem of F-valued integrals

In this section, the first is defining fuzzy-valued function, and its measurablity and integrablity, then we define F-valued integrals, most of these are adopt from Puri and Ralescu[4]. At last, Fubini theorem is given.

Symbol E^n denotes the set $\{u; R^n \to [0, 1] ; [u]^\alpha \in coR^n$ for all $\alpha \in (0, 1]\}$, where $[u]^\alpha = \{v \in R^n ; u(v) \geq \alpha\}$ The elements in E^n are called fuzzy numbers. A fuzzy-valued function is a mapping $F ; X \to E^n$. For a fuzzy-valued function F, the mapping $F_\alpha ; X \to coR^n ; x \to F_\alpha(x) = [F(x)]^\alpha$ is called α-cutting function, where $\alpha \in (0, 1]$

A fuzzy-valued function is Σ-measurable if its every α-cutting functions are Σ-measurable.

A Σ-measurable fuzzy-valued function $F ; X \to E^n$ is μ-integrably bounded if there exists a μ-integrable function $h: X \to R^1$ s.t

$$\sup_{\alpha \in (0,1]} \sup_{v \in F_\alpha(x)} \|v\| \leq h(x) \qquad \text{for all } x \in X.$$

The F-valued integral is defined as ;

$$(\int_X F(x) d\mu)(v) = \sup \{ \alpha \in (0,1] ; v \in \int_X F_\alpha(x) d\mu \}$$

for $v \in R^n$

Remark. The integral is called expected value by Puri and Ralescu [4]

Theorem 3.1 If a fuzzy-valued function $F: X \to E^n$ is μ-integrably bounded, then

(i) $\int_X F(x) d\mu \in E^n$

(ii) $[\int_X F(x) d\mu]^\alpha = \int_X F_\alpha(x) d\mu$ for $\alpha \in (0,1]$

The Proof of (i) is obvious, and (ii) can be proved similarly to the proof of Th 3.1 in [4]

Remark. All the concepts and results above are suitable for (Y, Σ', μ') and $(X \times Y, \Sigma \times \Sigma', \mu \times \mu')$

Next we give the main result of this paper.

Theorem 3.2 (Fubini th.) If a fuzzy-valued function $F: X \to E^n$ is $\mu \times \mu'$-integrably bounded, then

(i) the fuzzy-valued function $F(\cdot, y): x \to F(x, y)$ is μ-integrably bounded for $y \in Y$ μ'-a.e;

(ii) the fuzzy-valued function $I(\cdot): y \to I(y) = \int_X F(x, y) d\mu$ is μ'-integrably bounded, and

$$\int_{X \times Y} F(x, y) d\mu \times \mu' = \int_Y I(y) d\mu'.$$

The proof can be easily obtained by theorem 3.1 and theorem

2, 1 (Q.E.D)

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