Pan-Weak Convergence and Uniform Convergence Over the Class of Convex Sets of Fuzzy Measure Sequences

Ha Minghu and Wang Xizhao
Department of Mathematics, Hebei University, Baoding, Hebei, China.

Abstract

In this paper, we shall introduce the concept of metric pan-space and the pan-weak convergence of a type of fuzzy measure sequences and discuss the relation between the pan-weak convergence and the uniform convergence over the class of convex sets on R^k of the fuzzy measure sequences.

1. Introduction

Since Sugeno[1] introduced the fuzzy measure and the fuzzy integral in 1974, many articles [2—6] had been written dealing with differnet generalization of the fuzzy measure and integral,studying the structure of the fuzzy measure and discussing the covergence of a sequence of the fuzzy integrals for a fixed fuzzy measure, but almost no papers appeared discussing the sequence of fuzzy measures. The purpose of the present paper is to introduce a kind of convergence (called pan-weak convergence) of some type of fuzzy measure sequences and discuss the properties of the pan-weak convergence and the relation between the pan-weak convergence and the uniform convergence over the class of convex sets of the fuzzy measure sequences.

2. Metric pan-space, and the pan-weak convergence of a type of fuzzy measure sequences

Let (X,d) be a metric space. The smallest σ-algebra generated by all of open subsets of X is denoted by B(X). It is easy to see that B(X) is also the smallest σ-algebra generated by all of closed subsets of X.

Let (X,d) be a metric space, $(X,B(X),\mu)$ be a fuzzy measure space[2] and $A \in B(X)$. If

$$\mu(A)=\mathrm{Sup}[\mu(F): F\subset A,\ F \text{ is closed}]=\inf[\mu(G): G\supset A,\ G \text{ is open}]$$

Then A is called μ-regular. μ is called regular if A is μ-regular for every $A \in B(X)$.

Proposition 2.1 The fuzzy measure μ has the following property:

$$\mu(\underline{\mathrm{Lim}}_n A_n) \leq \underline{\mathrm{Lim}}_n \mu(A_n), \quad \overline{\mathrm{Lim}}_n \mu(A_n) \leq \mu(\overline{\mathrm{Lim}}_n A_n)$$

Where A_n $(n=1,2,\ldots) \in B(X)$, $\underline{\mathrm{Lim}}_n A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k$ and $\overline{\mathrm{Lim}}_n A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k$

The validity of the following two propositions is clear.

Proposition 2.2 Let (X,d) be a metric space and $(X,B(X),\mu)$ be a fuzzy measure space, $A \in B(X)$. Then A is μ-regular if and only if there exist a open subset G_r and a closed subset F_r such that

$$F_r \subset A \subset G_r \quad \text{and} \quad \mu(G_r) < \mu(F_r)+r \quad \text{for every } r>0.$$

Proposition 2.3 Let (X,d) be a metric space and $(X,B(X),\mu)$ and $(X,B(X),\nu)$ be two regular fuzzy measure spaces. If

$$\mu(F)=\nu(F) \quad (\text{or } \mu(G)=\nu(G))$$

for every closed subset F (or open subset G) then $\mu=\nu$.

Proposition 2.4 Let (X,d) be a metric space, E and F be two closed subsets in X and $E\cap F=\phi$. Then there exists a continuous function f defined in X such that

(1) $0\leq f(x)\leq 1$ for every $x \in X$.

(2) $f(x)=1$ if $x \in E$ and $f(x)=0$ if $x \in F$.

Proof. Define $d(x,F)=\inf[d(x,y): y\in F]$, $x\in X$, for every fixed F. It is easy to see that $d(x,F)$ is a uniformly continuous function on X. Since E and F are two disjoint closed subsets then

$$d(x,E)+d(x,F)>0 \quad \text{for every } x\in X.$$

Let

$$f(x) = \frac{d(x,E)}{\quad}$$

$$\overline{d(x,E)+d(x,F)}$$

It is easy to verify (1) and (2). The continuity of the function f can be obtained by the continuity of d(x,F) and d(x,E). The proof is completed.

Definition Let (X,d) be a metric space, $(X,B(X),\mu)$ be a fuzzy measure space, $R^+=[0,\infty)$ and # be a binary operation on R^+. Then $(X,d,B(X),\mu,R^+,\#)$ is called metric pan-space if

(a1) $a\#b=b\#a$ (a2) $a\#(b\#c)=a\#(b\#c)$

(a3) $a\#0=a$ (a4) $(a\#b)c=ac\#bc$

(a5) $\mathrm{Lim}(a_n\# b_n)=\mathrm{Lim}a_n\#\,\mathrm{Lim}b_n$ when $\mathrm{Lim}a_n$ and $\mathrm{Lim}b_n$ exist.

(a6) $a_1\# a_2\le b_1\# b_2$ when $a_1\le b_1$ and $a_2\le b_2$.

for all $a,b,c,a_i,b_i\,(i=1,2,\ldots)\in R^+$.

Remark. In the definition, we can take $a\#b=a+b$ or $a\#b=a\vee b=\max(a,b)$ It is easy to see that (a1)—(a6) hold. The operator $\vee$ in fuzzy mathematics has the same importance as the operator + in classical mathematics. Using the concept of pan-space, we can unite these two operations into the operator # .

Let $(X,d,B(X),\mu,R^+,\#)$ be a metric pan-space. We say μ is pan-additive if $\mu(E\cup F)=\mu(E)\#\mu(F)$ for all $E,F\in B(X)$, $E\cap F=\phi$. For every fixed metric space (X,d) and binary operation # , let

$$\Pi=\left\{\mu \,\middle|\, \begin{array}{l}\mu \text{ is regular, finite,pan-additive fuzzy measure on } B(X)\\ \text{such that } (X,d,B(X),\mu,R^+,\#) \text{ forms a matric pan-space}\end{array}\right\}$$

$$M^+=\left\{f \,\middle|\, \begin{array}{l}f \text{ is nonegative measurable function on } (X,B(X)) \text{ i.e.}\\ f^{-1}(A)\in B(X) \text{ for exery Borel subset on real line } A\end{array}\right\}$$

$$C^+=\left\{f \,\middle|\, f\in M^+,\ f \text{ is bounded and continuous}\right\}$$

For every $f\in M^+$ and $\mu\in\Pi$ let

$$(P)\int f d\mu=\mathrm{Lim}_{n\to\infty}\Big[\mathop{\#}_{m=1}^{n\cdot 2^n}\Big(\frac{m}{2^n}\mu(A_{mn})\Big)\Big] .$$

Then $(P)\int f d\mu$ is called pan-integral of the function f with respecte to μ [6,7].

Where
$$A_{mn}=\left\{x \mid \frac{m}{2^n}<f(x)\leq \frac{m+1}{2^n}\right\}$$

The following proposition is the simplest property of pan-integral.

Proposition 2.5 [6,7] If $f,g \in M^+$, $f\leq g$, $E\in B(X)$ then

$$(P)\int f d\mu \leq (P)\int g d\mu \quad \text{and} \ (P)\int I_E \, d\mu = \mu(E).$$

Where $I_E(x)$ is the characterastic function of the subset E.

Definition. Given μ, μ_n $(n=1,2,\ldots)\in \Pi$, if

$$\operatorname*{Lim}_{n}(P)\int f d\mu_n = (P)\int f d\mu \quad \text{for every } f \in C^+$$

then we say that the measure sequence $\{\mu_n\}$ pan-weak converges to μ denoted by $\mu_n \to \mu$ (PW), $(n\to\infty)$.

The following theorem shows that the definition is reasonable.

Theorem 2.6 Given $\mu, \nu \in \Pi$ if

$$(P)\int f d\mu = (P)\int f d\nu \quad \text{for every } f\in C^+ \quad \text{then} \quad \mu=\nu.$$

Proof. Let F be an arbitrary closed set in X

$$G_n=\left\{x: d(x,F)<\frac{1}{n}\right\} \ n=1,2,\ldots$$

Then $F\cap G_n^c=\phi$ for every integer n and $\inf[d(x,y): x\in F,\ y\in G_n^c]\geqslant \frac{1}{n}$
By proposition 2.4, there exists $f_n\in C^+$ such that $0\leq f\leq 1$, $f=1$ if $x\in F$ and $f=0$ if $x\in G_n^c$ for every integer n. It is easy to see

$$0\leq I_F\leq f_n\leq I_{G_n}$$

By proposition 2.5 we have

$$\mu(F)=(P)\int I_F d\mu \leq (P)\int f_n d\mu = (P)\int f_n d\nu \leq (P)\int I_{G_n} d\nu = \nu(G_n).$$

Take $n\to\infty$, by the continuity of fuzzy measure we have $\mu(F)\leq \nu(F)$.
The inequality $\nu(F)\leq \mu(F)$ can be obtained by interchanging ν

and μ in above argument.The proof is completed by proposition 2.3.

Theorem 2.7 If μ, $\mu_n(n=1,2,\ldots)\in\Pi$, and $\mu_n\to\mu$ (PW) then

$$\overline{\text{Lim}}\,\mu_n(F)\leq\mu(F) \quad\text{and}\quad \underline{\text{Lim}}\,\mu_n(G)\geq\mu(G)$$

Where F and G are arbitrary closed and open set in X respectively.

Proof. We only prove the first inequality. The second one can be obtained similarly.

Let F be an arbitrary closed set and $G_k=\{x:d(x,F)<\frac{1}{k}\}$, $k=1,2,\ldots$

Then $F=\bigcap_{k=1}^{\infty}G_k$, $F\cap G_k^c=\phi$ and $\inf[d(x,y):x\in F,\ y\in G_k^c]\geq\frac{1}{k}$.

By proposition 2.4 there exists $f_k\in c^+$ such that

$0\leq f_k\leq 1$ and $f_k(x)=1$ if $x\in F$ and $f_k(x)=0$ if $x\in G_k^c$ $k=1,2,\ldots$

It is easy to see $I_F\leq f_k=I_{G_k}$. Since $\mu_n\to\mu$ (PW) then

$$\overline{\underset{n}{\text{Lim}}}\,\mu_n(F)=\overline{\underset{n}{\text{Lim}}}(P)\int I_F\,d\mu_n\leq \text{Lim}(P)\int f_k\,d\mu_n=(P)\int f_k\,d\mu\leq(P)\int I_{G_k}d\mu=\mu(G_k)$$

for every integer k.

Take $k\to\infty$, since $G_k\searrow F$, we obtain $\overline{\text{Lim}}\,\mu_n(F)\leq\mu(F)$. The proof is completed.

Let $(X,d,B(X),\mu,R^+,\#)$ be a metric pan-space, $A\in B(X)$. We say A is μ-continuous if

$$\mu(A^\circ)=\mu(A)=\mu(\bar{A})$$

Where $A^\circ=\bigcup\{G|G\subset A,\ G\text{ is open}\}$ $\bar{A}=\bigcap\{F|F\supset A,\ F\text{ is closed}\}$.

Theorem 2.8 If μ, $\mu_n(n=1,2,\ldots)\in\Pi$ and $\mu_n\to\mu$ (PW), A is μ-continuous then $\lim\limits_{n\to\infty}\mu_n(A)=\mu(A)$

Proof. From theorem 2.7 and $\mu(A^\circ)=\mu(A)=\mu(\bar{A})$, we have

$$\mu(A)=\mu(A^\circ)\leq\underset{n}{\underline{\text{Lim}}}\,\mu_n(A^\circ)\leq\underset{n}{\underline{\text{Lim}}}\,\mu_n(A)\leq\overline{\underset{n}{\text{Lim}}}\,\mu_n(A)$$

$$\leq\overline{\underset{n}{\text{Lim}}}\,\mu_n(\bar{A})\leq\mu(\bar{A})=\mu(A)$$

It implies $\lim_{n\to\infty}\mu_n(A)=\mu(A)$.

3. Some results on the conex sets

Let R^k be k-dimensional Euclidean space, ρ be a metric on R^k, $A\subset R^k$. We say A is convex if $rx+(1-r)x\in R^k$ for every $r\in[0,1]$ and $x,y\in A$.

Convex sets possess the following simple properties.

Let $\{A_\gamma\}$ be a class of convex sets ($r\in\Gamma$). Then

(1) $\bigcap_{\gamma\in\Gamma}A_\gamma$ is a convex set.

(2) $\bigcup_{\gamma\in\Gamma}A_\gamma$ is a convex set if $\{A_\gamma\}$ is a chain.

(3) For every $r\in\Gamma$, $(\overline{A^\circ})=\bar{A}$, $(\bar{A})^\circ=A^\circ$.

Let $A^r=\{x\in R^k \mid \rho(x,A)<r\}$ for $r>0$. Then

(4) A is convex if A is convex.

Let A and B be two convex sets on R^k, $A^\circ\neq\phi$ and $A^\circ\cap B=\phi$. Then

(5) there exists a hyperplane H to keep A separate from B.

In the following discussion, we suppose

$$\mathcal{A}=\{A \mid A\subset R^k, A \text{ is bounded, closed and convex}\}$$
$$S_r=\{x \mid x\in R^k,\ \rho(x,0)\leq r\} \qquad (r>0)$$

Define $\delta(A,B)=\inf[r: A^r\supset B,\ r>0]$ and $\delta=\inf[r: A\supseteq B,\ B^r\supseteq A, r>0]$ for $A,B\in\mathcal{A}$ Let $\Delta(A,B)=\delta(A,B)+\delta(B,A)$, then $\Delta(A,B)$ defines a metric on and $(\mathcal{A},\Delta)$ is a metric space. Let $d(A,B)=\delta$. Then $(\mathcal{A},d)$ is also a metric space. It is easy to prove that the metric Δ and the metric d are equivalent.

Theorem 3.1 (Blaschke,[8]) The all of closed convex sets in $S_r(r>0)$ compose a compact metric space i.e. for every sequence of nonempty , closed and convex sets in S_r denoted by $\{C_n\}$, there exists a subsequence $\{C_{n_j}\}$ converging to a closed and convex set C and

$$C=\bigcap_{k=1}^{\infty}\overline{\bigcup_{j=k}^{\infty}C_{n_j}}$$

Theorem 3.2 Let $\{C_n\}$ be a sequence of uniformly bounded, closed and convex sets. If there exists $C\in\mathcal{A}$ and $C^\circ\neq\phi$ such that $d(C_n,C)\to 0$ $(n\to\infty)$, then there exists N such that $C_n^\circ\neq\phi$ for $n\geq N$ and

$$C\subset\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}C_k^\circ$$

Proof. We first prove $C_n^\circ \neq \phi$ for $n \geqslant N$. If it is not true then there exists a subsequence of $\{C_n\}$, denoted still by $\{C_n\}$, such that $C_n^\circ \neq \phi$ for all n. Hence there exists a hyperplan H_n for every n such that $C_n \subset H_n$. Since $C^\circ \neq \phi$, then there exists a open sphere $B(z,r)$ and a one-dimensional affine subspace M_n passing through z such that $M_n \perp H_n$.

Let $$y_n = M_n \cap (\overline{B(z,r)} - B(z,r)).$$
Then $$d(C_n,C) \geqslant \delta(C_n,C) \geqslant \delta(C_n,B(z,r)) \geqslant \rho(y_n,H_n) \geqslant r > 0$$
This contradicts with $d(C_n,C) \to 0$ $(n \to \infty)$. Hence there exists N such that $C_n^\circ \neq \phi$ for $n \geqslant N$.

In the following we prove $C^\circ \subset \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty C_k^\circ$

We can suppose $C_k^\circ \neq \phi$ for every k. If $z \in C$ but $z \overline{\in} \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty C_k^\circ$ then there exists $k_n \geqslant n$ for every n such that $z \overline{\in} C_{k_n}^\circ$. Since $z \in C^\circ$ and the boundary of C (denoted by D(C)) is closed, there exists $w \in D(C)$ such that $$\rho(z,w) = \delta = \inf_{y \in D(C)} \rho(z,y) > 0.$$

Notice $z \overline{\in} C_{k_n}^\circ$, $\{z\} \cap C_{k_n}^\circ = \phi$, $C_{k_n}^\circ \neq \phi$, then there exists a hyperplane H_{k_n} to keep $\{z\}$ separate from C_{k_n} that implies $d(C_{k_n},C) > \delta$. This contradicts with $d(C_n,C) \to 0$ $(n \to \infty)$ that completes the proof.

From theorem 3.1 and 3.2 we can easily obtain the following

Theorom 3.3 Let $\{C, C_n, n=1,2,\dots\}$ be a sequence of uniformly bounded and closed convex sets and $C^\circ \neq \phi$. If $d(C_n,C) \to 0$ $(n \to \infty)$ then
$$C = \overline{\bigcup_{n=1}^\infty \bigcap_{k=n}^\infty C_k^\circ} = \overline{\bigcup_{n=1}^\infty \bigcap_{k=n}^\infty C_k} = \bigcap_{n=1}^\infty \overline{\bigcup_{k=n}^\infty C_k}$$

4. Uniform convergence of fuzzy measure sequences over the class of convex sets on R^k.

In this section, the convex sets considered are always measurable convex sets on R .

Lemma Let $\mu \in \Pi$, $\{C_n, C\} \subset \mathcal{A} \cap S_m$ for $m > 0$. If $d(C_n,C) \to 0$ $(n \to \infty)$ and every convex set is μ-continuous then $\mu(C_n) \to \mu(C)$ $(n \to \infty)$.

Proof. If $C^o=\phi$ then $\mu(C)=0$ because C is μ-continuous. Hence from Prop.2.1 and theorem 3.1 we have

$$0\leq\overline{\mathrm{Lim}}_n\,\mu(C_n)\leq\mu(\overline{\mathrm{Lim}}_n C_n)=\mu(\cap_{k=1}^{\infty}\cup_{n=k}^{\infty}C_n)\leq\mu(\cap_{k=1}^{\infty}\overline{\cup_{n=k}^{\infty}C_n})=\mu(C)=0$$

That implies $\mu(C_n)\to\mu(C)$ $(n\to\infty)$.

If $C^o\neq\phi$ then, from section 3 and prop.2.1 we have

$$\mu(C)=\mu(C^o)\leq\mu(\cup_{n=1}^{\infty}\cap_{k=n}^{\infty}C_k^o)=\mu(\underline{\mathrm{Lim}}_k C_k^o)\leq\underline{\mathrm{Lim}}_k\,\mu(C_k^o)\leq\underline{\mathrm{Lim}}_k\,\mu(C_k)$$

and

$$\mu(C)=\mu(\cap_{n=1}^{\infty}\overline{\cup_{k=n}^{\infty}C_k})\geq\mu(\cap_{n=1}^{\infty}\cup_{k=n}^{\infty}C_k)=\mu(\overline{\mathrm{Lim}}_k C_k)\geq\overline{\mathrm{Lim}}_k\,\mu(C_k)$$

That implies $\mu(C_k)\to\mu(C)$ $(k\to\infty)$. The proof is completed.

Note 1. If C_n is not closed but $d(\overline{C_n},C)\to 0$ $(n\to\infty)$ then we still have $\mu(C_n)\to\mu(C)$ $(n\to\infty)$.

Theorem 4.1 Let $\{\mu,\mu_n\}\subset\Pi$, $\{C,C_n\}\subset\mathscr{A}\cap S_m$ for some $m>0$. If $\mu_n\to\mu$ (PW), $d(C_n,C)\to 0$ and every convex set is μ-continuous then $\mu_n(C_n)\to\mu(C)$ and $\mu_n(C_n^o)\to\mu(C^o)$ $(n\to\infty)$.

Proof. Let $d(C_n,C)=d_n$ $n=1,2,\ldots$ and $d'_N=\sup\{d_n\mid n\geq N\}$ $N=1,2,\ldots$ Then $d'_N>d$ and $d'_N\downarrow 0$. $(N\to\infty)$.

From $d(C_n,C)=d_n$ we obtain $C^{d_n+\frac{1}{n}}\supset C$. Hence, for every fixed N and $n>N$, $d'_N+\frac{1}{N}\geq d_n+\frac{1}{n}$ holds. So

$$\mu_n(C_n^o)\leq\mu_n(C_n)\leq\mu_n(C^{d_n+\frac{1}{n}})\leq\mu_n(C^{d'_N+\frac{1}{N}}).$$

Notice $\mu_n\to\mu$ (PW) and every convex set is μ-continuous, we have

$$\overline{\mathrm{Lim}}_n\,\mu_n(C_n^o)\leq\overline{\mathrm{Lim}}_n\,\mu_n(C_n)\leq\overline{\mathrm{Lim}}_n\,\mu_n(C^{d_n+\frac{1}{n}})\leq\overline{\mathrm{Lim}}_n\,\mu_n(C^{d'_N+\frac{1}{N}})=\mu(C^{d'_N+\frac{1}{N}})$$

$$=\mu(\overline{C^{d'_N+\frac{1}{N}}})\qquad\ldots\ldots(4.1)$$

Let $N\to\infty$. Then we have $d(\overline{C^{d'_N+\frac{1}{N}}},C)\to 0$.

From the lemma, we know that

$$\mathrm{Lim}_N\,\mu(C^{d'_N+\frac{1}{N}})=\mu(C).$$

let $N \to \infty$ in (4.1), then

$$\overline{\lim_{n}}\, \mu_n(C_n^\circ) \le \overline{\lim_{n}}\, \mu_n(C_n) \le \mu(C) \quad \ldots\ldots(4.2)$$

If $C^\circ = \phi$, then from (4.2)

$$\mu_n(C_n^\circ) \to 0 \text{ and } \mu_n(C_n) \to 0 \text{ hold } (n \to \infty).$$

If $C^\circ \neq \phi$, then for fixed N and n N, we have

$$\underline{\lim_{n}}\, \mu_n(C_n) \geqslant \underline{\lim_{n}}\, \mu_n(C_n^\circ) \geqslant \underline{\lim_{n}}\, \mu_n(\bigcap_{K=n}^{\infty} C_K) \geqslant \underline{\lim_{n}}\, \mu_n(\bigcap_{K=N}^{\infty} C_K^\circ)$$

$$= \mu(\bigcap_{K=N}^{\infty} C_K^\circ) \geqslant \mu(\bigcap_{K=N}^{\infty} C_K)^\circ = \mu(\bigcap_{K=N}^{\infty} C_K) \quad \ldots\ldots (4.3)$$

Let $N \to \infty$ in (4.3) , from theorem 3.3 we obtain

$$\underline{\lim_{n}}\, \mu_n(C_n) \geqslant \underline{\lim_{n}}\, \mu_n(C_n^\circ) \geqslant \lim_{N} \mu(\bigcap_{K=N}^{\infty} C_K) = \mu(\bigcup_{N=1}^{\infty} \bigcap_{K=N}^{\infty} C_K)$$

$$= \mu(\overline{\bigcup_{N=1}^{\infty} \bigcap_{K=N}^{\infty} C_K}) = \mu(C)$$

The proof is completed by combining (4.2).

Note 2. If C_n is not closed but $d(C_n, C) \to 0$ $(n \to \infty)$ then $\mu_n(C_n) \to \mu(C)$ $(n \to \infty)$ also holds.

Theorem 4.2 Let $\{\mu, \mu_n\} \subset \Pi$, $\mu_n \to \mu$ (PW). $\mathcal{A}_m$ denotes the all of convex sets in S_m and every set in S_m is μ-continuous. Then μ_n converges to μ uniformly on $\mathcal{A}_m$. i.e. $\sup_{C \in \mathcal{A}_m} |\mu_n(C) - \mu(C)| \to 0$.

Proof. If it is not true then there exists $\varepsilon_0 > 0$ and a sequence of convex sets $\{C_K\} \subset \mathcal{A}_m$ such that $|\mu_K(C_K) - \mu(C_K)| \geqslant \varepsilon_0$ ......(4.4)
Obviously, we can suppose $C_K^\circ \neq \phi$ for every k. By Blachke theorem, we know that there exists a subsequence of $\{\overline{C}_K\}$, denoted still by $\{\overline{C}_K\}$, and a nonempty closed set $C \in \mathcal{A}_m$ such that $d(\overline{C}_K, C) \to 0$ $(k \to \infty)$
By the lemma, we obtain $\mu(C_K) \to \mu(C)$ $(k \to \infty)$
By theorem 4.1, we obtain $\mu_K(C_K) \to \mu(C)$ $(k \to \infty)$.
Hence, $|\mu_K(C_K) - \mu(C_K)| \le |\mu_K(C_K) - \mu(C)| + |\mu(C_K) - \mu(C)| \to 0$ $(k \to \infty)$
That contradicts with (4.4). The proof is completed.

Theorem 4.3 Let $\{\mu, \mu_n\} \subset \Pi$, $\mu_n \to \mu$ (PW), $\mathcal{B}_m = \mathcal{A} \cap S_m$ and every set in $\mathcal{B}_m$ is μ-continuous. Then, for every $\varepsilon > 0$, there exists $\delta=\delta(\varepsilon)>0$ and $N=N(\varepsilon)$, such that

$$\sup_{C\in\mathcal{B}_m} |\mu_n(C)-\mu(C^{\delta})|<\varepsilon \qquad \text{when } n\geqslant N.$$

Proof. We first prove $\mu_n(C)\leq \mu(C^{\varepsilon})+\varepsilon \quad (n\geqslant N)$ ...... (4.5)
If (4.5) is not true then there exists $\varepsilon_0>0$ and a sequence of convex sets $\{C_n\}\subset \mathcal{B}_m$ such that

$$\mu_n(C_n)>\mu(C_n^{\varepsilon_0})+\varepsilon_0 \qquad \text{......(4.6)}$$

By Blachke theorem, there exists a subsequence of $\{C_n\}$, denoted still by $\{C_n\}$ and $C\in\mathcal{B}_m$ such that $d(C_n, C)\to 0$ $(n\to\infty)$. By theorem 30 (see[8]) we obtain
$d(\overline{C_n^{\varepsilon_0}},\overline{C^{\varepsilon_0}})\to 0$. By the lemma, we have $\mu(C_n^{\varepsilon_0})\to\mu(\overline{C^{\varepsilon_0}})=\mu(C^{\varepsilon_0})$ $(n\to\infty)$.
By theorem 4.1, we obtain

$$\mu_n(C_n)\to\mu(C) \quad (n\to\infty)$$

Let $n\to\infty$ in (4.6), we have

$$\mu(C)\geqslant \mu(C^{\varepsilon_0})+\varepsilon_0 \qquad \text{... ...(4.7)}$$

(4.7) is a contradiction because μ is finite and $\varepsilon_0>0$. That is to say that (4.5) is true.

In the following, we prove $\mu(C^{\delta})-\varepsilon\leq\mu_n(C)$ $(n\geqslant N)$ ......(4.8)
If (4.8) is not true then there exists $\varepsilon_1>0$ and $C_k\in\mathcal{B}_m$ $(k=1,2,\dots)$ such that

$$\mu(C_k^{\frac{1}{k}})-\varepsilon_1>\mu_k(C_k) \qquad \text{... ...(4.9)}$$

By Blachke theorem, there exists $C\in\mathcal{B}_m$ such that

$d(\overline{C_k^{\frac{1}{k}}},C)\to 0$ $(k\to\infty)$. Since $d(C_k,C)\leqslant d(\overline{C_k^{\frac{1}{k}}},C_k)+d(C,\overline{C_k^{\frac{1}{k}}})\to 0$

$(k\to\infty)$, we obtain $\mu_k(C_k^{\frac{1}{k}})\to\mu(C)$ and $\mu(C_k)\to\mu(C)$ $(k\to\infty)$ from theorem 4.1 and the lemma.

Let $k\to\infty$ in (4.9), we have $\mu(C)-\varepsilon_1\geqslant\mu(C)$ ... ...(4.10)
Because $\varepsilon_1>0$, (4.10) is a contradiction. Hence, (4.8) is true.
Combining (4.5) with (4.8), we complete the proof.

References

[1] M.Sugeno, Theory of fuzzy integrals and its application, Ph. D. Dissertation, Tokyo Institute of Technology (1974).

[2] D.Ralescu and G.Adams, The fuzzy integral, J. Math. Anal.

Appl. 75(1980) 562-570.
[3] Wang Zhenyuan, The autocontinuity of set function and the fuzzy integral, J. Math. Anal. Appl. 99(1984) 195-218.
[4] R.Kruse, On the construction of fuzzy measures, Fuzzy Sets and Systems 8(1982), 323-327.
[5] Wang Zhenyuan, Asymptotic structure characteristics of fuzzy measure and its applications, Fuzzy Sets and Systems 16(1985) 277-290
[6] Yang Qingji and Song Renming, The Pan-integral on the fuzzy measure space, Fuzzy Mathematics (China)3 (1985) 277-290
[7] Wang Xizhao and Ha Minghu, Pan-fuzzy integral, BUSEFAL 43 (1990), 37-41
[8] Eggleston, Convexity, Combridge tracts in mathematics and mathematical physics No.47, Cambridge Univ. Press(1958)
[9] Probability and Measure, (in Chinese), Zhong Shan Univ. Press(1982).