RELATIONS AND DISTINCTIONS BETWEEN POSSIBILITY MEASURES AND FUZZY MEASURES

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Abstract In this paper, we discuss some relations and distinctions between possiblity measure and F-additive fuzzy measure, F-additive semi-continuous fuzzy measure which all defined on the power set of X with card(X) $\leq \aleph_1$.

Keywords: fuzzy measure, possibility measure, F-additivity

1. Introduction

The concepts of possibility measure and fuzzy measure were firstly introduced by L. A. Zadeh[1] and M. Sugeno[2] respectively. Many authors(A. Kandel[3], H. T. Nguyen[4] etc) pointed out that a possibility measure is a fuzzy measure before M. L. Puri[5] and Z. Wang[6] showed that the above statement is not ture in general when the discourse set X is infinite. Following [5] and [6], in this paper, the relations and distinctions between possibilioty measure, F-additive fuzzy measure and F-additive semi-continuous fuzzy measure([6]) are discussed in detail for the cases that card(X) $\leq \aleph_1$. We show that the concept of possibility measure is weaker than the one of F-additive fuzzy measure and stronger than the one of F-additive semi-continuous fuzzy measure.

2. Definitions and assumptions

Let X be a non-empty set and $\mathcal{A}$ a σ-algebra of subsets of X. A fuzzy measure is a set function μ: $\mathcal{A} \rightarrow [0, 1]$ with the properties:

(FM1) $\mu(\emptyset) = 0$;

(FM2) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$;

(FM3) $A_n \in \mathcal{A}$, $A_n \uparrow \Rightarrow \mu(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$;

(FM4) $A_n \in \mathcal{A}$, $A_n \downarrow \Rightarrow \mu(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

When μ satisfies (FM1), (FM2) and (FM3), we call μ a semi-continuous fuzzy measure.

Let $P(X) = \{A;\ A \subset X\}$, A possibility measure is a set function μ: $P(X) \rightarrow [0, 1]$ with the properties:

(P1) $\mu(\emptyset) = 0$;

(P2) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$;

(P3) $\mu(\bigvee_{t\in T} A_t) = \bigvee_{t\in T} \mu(A_t)$, where T is an arbitary index set.

We can easily prove thew following two propositions.

Proposition1 μ is a possibility measure iff μ satisfies (P1), (P2) and

(P3)' $\mu(A) \leq \bigvee_{x\in A} \mu(\{x\}) \qquad \forall A \in P(X)$

Proposition2 A fuzzy measure or a semi-continuous fuzzy measure μ is a possibility measure iff μ satisfies (P3)'.

As a possibilitry measure μ is defined on P(X) and has F-additivity, i. e. $\mu(A \cup B) = \mu(A) \vee \mu(B)$, then if we want to discuss the relations and distinctions between possibility measure, fuzzy measure, semi-continuous fuzzy measure, we must assume that $\mathcal{A} = P(X)$ and fuzzy measure and semi-continuous fuzzy measure have F-additivity.

Throughout this paper, we assume that the fuzzy measure and semi-continuous fuzzy measure are defined on $P(X)$ and have F-additivity, and note

$FM = \{\mu;\ \mu$ is a F-additive fuzzy measure on $P(X)\}$

$PM = \{\mu;\ \mu$ is a possibility measure on $P(X)\}$

$SCFM = \{\mu;\ \mu$ is a F-additive semi-continuous fuzzy measure on $P(X)\}$

3. The relations between FM, PM and SCFM

3.1 X is an arbitary set

Theorem1 $FM \subset SCFM$, $PM \subset SCFM$.

Proof obvious.

3.2 $Card(X) < \aleph_0$

Let $X = \{x_1, x_2, \cdots, x_n\}$, we have

Theorem2 $FM = PM = SCFM$

Proof. In [5] and [6], it is pointed out that $PM \subset FM$, then, from Theorem1, we only need to prove that $SCFM \subset PM$.

Let $\mu \in SCFM$, then, $\forall A \in P(X)$, $A = \{x_{i_1}, x_{i_2}, \ldots, x_{i_K}\}$ ($k \leq n$), we have, by using the F-additivity of μ, that

$$\begin{aligned}\mu(A) &= \mu(\{x_{i_1}, x_{i_2}, \ldots, x_{i_K}\}) \\ &= \mu(\{x_{i_1}\}) \vee \mu(\{x_{i_2}, \ldots, x_{i_K}\}) \\ &= \mu(\{x_{i_1}\}) \vee \mu(\{x_{i_2}\}) \vee \mu(\{x_{i_3}, \ldots, x_{i_K}\}) = \ldots \\ &= \mu(\{x_{i_1}\}) \vee \mu(\{x_{i_2}\}) \vee \ldots \vee \mu(\{x_{i_k}\}) = \bigvee_{x \in A} \mu(\{x\})\end{aligned}$$

Applying Proposition2, we know that $\mu \in PM$ and this ends the proof.

2.3 $Card(X) = \aleph_0$

Let $X = \{x_1, x_2, \ldots, x_n, \ldots\}$, we have

Theorem3 $FM \subset PM = SCFM$

Proof. From Theorem1, it is sufficent to prove that $SCFM \subset PM$.

Let $\mu \in$ SCFM, then, $\forall$ A $\in$ P(X), (1) if card(A) $< \aleph_0$, the conclusion follows by refering the proof of Theorem2; (2) if card(A) = $\aleph_0$, note A = $\{x_{i_1}, \ldots, x_{i_n}, \ldots\}$, there holds

$$\mu(A) = \mu(\{x_{i_1}, \ldots, x_{i_n}, \ldots\}) = \mu(\lim_{n\to\infty}\{x_{i_1}, \ldots, x_{i_n}\})$$
$$= \lim_{n\to\infty} \mu(\{x_{i_1}, \ldots, x_{i_n}\})$$
$$= \lim_{n\to\infty} \mu(\{x_{i_1}\}) \vee \cdots \vee \mu(\{x_{i_n}\}) \text{ (cf. the proof of Theorem2)}$$
$$= \bigvee_{n=1}^{\infty} \mu(\{x_{i_n}\}) = \bigvee_{x\in A} \mu(\{x\})$$

Using Proposition2, we obtaion that $\mu \in$ PM, and then the proof is complete.

2.4 $\aleph_0 <$ card(X) $\leq \aleph_1$

We have

Theorem4 FM $\subset$ PM $\subset$ SCFM

In the proof of Theorem4, we consider that Cantor's continuum hypothesis is true, that is to say, $\aleph_0 <$ card(X) $\leq \aleph_1$ is equivalent to that card(X) = $\aleph_1$. It must be pointed out that even we don't admit the continuum hypothesis, Theorem4 is also true, which wiil be explained at the end of this section.

Proof of Theorem4 We suppous that card(X) = $\aleph_1$.

It is obvious that P(X) has only the operations "$\bigcup$", "$\bigcap$" and "_" of subsets of X and has nothing to do with the structures (algebraic, topological or ordered etc)defined on X, it is equivalent to say that we can endow any structure on X and this action has no inflence on our problem. At the same time, since card(X) = $\aleph_1$, then there exists a bijection between X and interval [0, 1]. Then we may assume that X has the same structure of [0, 1] or we regard X = [0, 1] and prove Theorem4, this assumption is without any loss of generailiy to the proof. In the following, we assume that X = [0, 1].

From Theorem1, it only need to prove that FM $\subset$ PM.

Let $\mu \in$ FM, we show that $\mu \in$ PM.

$\forall A \in P([0, 1])$, if $\mu(A) = 0$, then

$$\mu(A) \leq \bigvee_{x \in A} \mu(\{x\})$$

and the conclusion follows.

If $\mu(A) > 0$, noting $A_1^1 = A \cap [0, 1/2]$, $A_2^1 = A \cap [1/2, 1]$, from F-additivity of μ, we have

$$\mu(A) = \mu(A_1^1) \vee \mu(A_2^1)$$

When $\mu(A) = \mu(A_1^1)$, take $A_1 = A_1^1$, otherwise, take $A_1 = A_2^1$, we have

$$\mu(A) = \mu(A_1)$$

Without any loss of generility, we suppose that $A_1 \subset [0, 1/2]$. Observing that $A_1^2 = A_1 \cap [0, 1/4]$, $A_2^2 = A_1 \cap [1/4, 1/2]$, similar to above discussion, we can choose $A_2 = A_1^2$(or A_2^2) such that

$$\mu(A_2) = \mu(A_1)$$

According to the same priciple, we can choose A_3, A_4, ..., A_n ..., such that $\mu(A_n) = \mu(A)$, $\forall n \in N$, and $A \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$

Furthermore, from the process of choosing $\{A_n\}$, we easily know that $\bigcap_{n=1}^{\infty} A_n = \emptyset$ or there exists unique $\tilde{x} \in [0, 1]$ such that $\bigcap_{n=1}^{\infty} A_n = \{\tilde{x}\}$.

As $\mu \in$ FM, we obtain that

$$\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mu(A_n) = \mu(A) > 0$$

This implies that $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$, i.e. $\bigcap_{n=1}^{\infty} A_n = \{\tilde{x}\}$ and hence

$$\mu(A) = \mu(\{\tilde{x}\}) \leq \bigvee_{x \in A} \mu(\{x\})$$

and Theorem4 is proved.

Remark. If we don't admit the continuum hypothesis, we can embed X into a subset of [0, 1](in the sense of above discussion) and prove Theorem4 according to the same process mentioned above.

4. Distinctions between FM, PM and SCFM

4.1 When card(X) $\geqq \aleph_0$, we have FM $\neq$ PM

We can define $\mu \in$ PM as follows:

$$\mu(A) = 1 \qquad A \neq \emptyset$$
$$\quad\;\; = 0 \qquad A = \emptyset$$

but $\mu \notin$ FM. In fact, as X is infinite, we can choose a contable subset $\{x_1, x_2, \ldots, x_n, \ldots\} \subset X$. Note $A_n = \{x_n, x_{n+1}, \ldots\}$, $\forall\, n \in N$, we have $A_n \downarrow \emptyset$, $\mu(A_n) = 1$, but $\mu(\emptyset) = 0$.

4.2 When card(X) $> \aleph_0$, we have PM $\neq$ SCFM

We can construct $\mu \in$ SCFM as follows:

$$\mu(A) = 1 \qquad \mathrm{card}(A) > \aleph_0,$$
$$\quad\;\; = 0 \qquad \mathrm{card}(A) \leq \aleph_0,$$

but $\mu \notin$ PM. In fact, $\mu(X) = 1$ and $\bigvee_{x \in X} \mu(\{x\}) = 0$.

5. Conclusions

The results arrived at in the section 3 and 4 can be summaried below:

(1) Card(X) $< \aleph_0$: $(\mu \in \mathrm{FM}) \rightleftarrows (\mu \in \mathrm{PM}) \rightleftarrows (\mu \in \mathrm{SCFM})$;

(2) Card(X) $= \aleph_1$: $(\mu \in \mathrm{FM}) \underset{\not\leftarrow}{\rightarrow} (\mu \in \mathrm{PM}) \rightleftarrows (\mu \in \mathrm{SCFM})$;

(3) $\aleph_0 <$ card(X) $\leq \aleph_1$: $(\mu \in \mathrm{FM}) \underset{\not\leftarrow}{\rightarrow} (\mu \in \mathrm{PM}) \underset{\not\leftarrow}{\rightarrow} (\mu \in \mathrm{SCFM})$.

where "$\longrightarrow$" and "$\not\longrightarrow$" mean "implies" and "does not implies" respectively.

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