FACTOR SPACES, SHEAFS AND CATEGORIES

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Abstract: In this paper, we study Topos properties of category **FRT(Y)**, which is based on thoughts of Factor Rattans. We also prove that a Factor Space can make a module sheaf.

Keywords: Factor Spaces, Categories, Topos, Sheafs, Rattans.

1. INTRODUCTION

Since L. A. Zadeh introduced concept of fuzzy sets, fuzzy set theory have been used in study of **AI**, controls and decision. However, knowledge representation has appeared to be more important in applications of these fields. The factor spaces provide an approach for knowledge representation .

A factor spaces is a family of sets $\{X_t: t\in L\}$ with index set L, a Boolean Algebra $L=(L,\wedge,\vee,c)$, satisfying

(1) $X_0=\phi$

(2) If $T\subseteq L$ is independent, i.e. for any t_1, $t_2\in T$, $t_1\wedge t_2=0$ whenever $t_1\neq t_2$ then $X_{\vee\{t|t\in T\}}=\prod_{t\in T} X_t$

where 0 and 1 are smallest and largest element of L respectively, and $\prod$ is Cartesian product operator .

We call t, $t\in L$, a factor, X_t the corresponding state space, and X_1, the whole state space, L factor set.

For example, concept 'old' can be described as a fuzzy subset A of the age-universe U, it also can be described as a fuzzy subset B of the face-universe U, and so on. Here 'age', 'face', …, are not objects itself. We call them factors. When t='sex ', X_t can be described as $X_t=\{male, female\}$.

In this paper, we will study Topos properties of category **FRT(Y)** and prove that a Factor Space can make a module sheaf.

2. RATTAN CATEGORY FRT(Y)

One of important concepts in Factor Spaces is Factor Rattan:

Let O be sets of some objects. For any object $a\in O$, Y_a is denoted as set of factors which are related to a.

Let $Y=\{t \mid \exists a\in O,\ s.t.\ t\in Y_a\}$, then we can introduce a binary relation R: $\forall t\in Y$, $R(a, t)$ represents that object a and factor t is related.

$$\text{Let } \xi:\ \mathcal{P}(O) \longrightarrow \mathcal{P}(Y)$$
$$A \longrightarrow \xi(A)=\{t\in Y \mid \forall a\in A, R(a, t)\}$$
$$\eta:\ \mathcal{P}(Y) \longrightarrow \mathcal{P}(O)$$
$$C \longrightarrow \eta(C)=\{a\in O \mid \forall t\in C, R(a, t)\}$$

Where $\mathcal{P}(O)=\{A \mid A\subseteq O\}$, $\mathcal{P}(Y)=\{C \mid C\subseteq Y\}$, then we have:

Proposition. (1) $A\supseteq B \Longrightarrow \xi(A)\subseteq\xi(B)$

(2) $\xi(A\cup B)=\xi(A)\cap\xi(B)$ for any $A, B\subseteq O$

(3) $C\supseteq D \Longrightarrow \eta(C)\subseteq\eta(B)$

(4) $\eta(C\cup D)=\eta(C)\cap\eta(D)$ for any $C, D\subseteq Y$

Definition 2.1 ξ, η as above is called as Factor Rattans.

$\xi(A)$ is called as factor set of A.

Remark: Factor Rattans indicate the following facts:

When set of some objects is divided into subsets, factor sets of corresponding subsets increase presently.

For example, A=set of all men and women, B= set of all men, C= set of all women, then $\xi(B)\supseteq\xi(A)$, $\xi(C)\supseteq\xi(A)$ and $\xi(A)=\xi(B)\cap\xi(C)$ (Since $A=B\cup C$)

Factor Rattan is a important concept in Factor Spaces. We are going to introduce a category **FRT(Y)** which is based on the Factor Rattans, and Topos properties of **FRT(Y)** are studied.

Definition 2.2 Let Y be a fixed set and for a set X, existing a mapping ξ: $\mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$, $A \longrightarrow \xi(A)$ satisfying:

(1) $A\supseteq B \Longrightarrow \xi(A)\subseteq\xi(B)$, (2) $\xi(A\cup B)=\xi(A)\cap\xi(B)$, for any $A,B\subseteq O$ then ξ is called as a Rattan over Y, which is denoted as (X,ξ).

Definition 2.3 Let **FRT(Y)** is a category, its objects are Rattans over Y; arrows from (X_1,ξ_1) to (X_2,ξ_2) are mapping f: $X_1 \longrightarrow X_2$ and satisfying: $\xi_2(f(A))\supseteq\xi_1(A)$, for any $A\subseteq X_1$. **FRT(Y)** is called as Rattan Category over Y.

Theorem 2.1: Category **FRT(Y)** satisfies all TOpos properties except one for it has no Sunobject Classifier.

3. Factor Spaces and Sheafs

Let $\{X_t | t\in L\}$ be a Factor Space, then L can be seen as a Boolean ring $L=(L,+,\cdot)$, where $s+t=(s\wedge t^c)\vee(s^c\wedge t)$, $st=s\wedge t$, $\forall s,t\in T$.

Let $X_s \dotplus X_t = X_{s+t}$, $sX_t = X_{st}$, then $(\{X_t | t \in L\}, \dotplus, \cdot)$ is a module over ring L.

Now Let J is a topology over L, and

$$Q_U = \{X_t \mid t \le U^{\vee}\} \text{ for any } U \in J, \text{ where } U^{\vee} = \vee\{h \mid h \in U\},$$

then we have

Proposition 3.1 Q_U is a module over L.

Proposition 3.2 γ_{UV}: $Q_U \longrightarrow Q_V$

$$X_t \longrightarrow X_{t \wedge U^{\vee}}$$

then γ_{UV} is a module homomorphism from Q_U to Q_V .

Theorem 3.1 If

$$(U \cap V)^{\vee} = U^{\vee} \wedge V^{\vee} \text{ , for any } U, V \in J \qquad (1)$$

Then $\{Q_U, \gamma_{UV}\}_{(U \in J)}$ is a module sheaf over (L, J)

Corollarly . If

$$t \wedge V^{\vee} = t \wedge (U \cap V)^{\vee} \text{ , for any } U, V \in J, \ X_t \in Q_U \qquad (2)$$

Then $\{Q_U, \gamma_{UV}\}_{(U \in J)}$ is a L-module sheaf.

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