Fuzzy Ideals Generated by Fuzzy Point in Semigroups

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Abstract

In this paper, fuzzy left (right) ideals generated by fuzzy point in semigroups are introduced. Some characteristics of them are given and some equivalent conditions of a fuzzy inverse subsemigroup are obtained.

Keywords : Fuzzy ideal generated by fuzzy point , Fuzzy inverse subsemigroup, Fuzzy regular element.

1. Introduction

In [2, 3, 4] , N. Kuroki introduced a fuzzy subsemigroup, a fuzzy left (right) ideal, a fuzzy ideal etc., [5] discussed a fuzzy regular subsemigroup, a fuzzy completely regular subsemigroup etc. and some characteristics of them are given. In this paper, the definitions of fuzzy left (right) ideals generated by fuzzy point, fuzzy inverse subsemigroups are given. Some characteristics are obtained and we discuss the properties of fuzzy regular elements.

2. Preliminaries

Throughout this paper, we always suppose that S is a semigroup. For any $x, y \in S$, $xy \in S$ means $x \cdot y \in S$, where "$\cdot$" is a binary operation in S. Define (see [1] , P4)

$$S^1 = \begin{cases} S & \text{if } 1 \in S, \\ S \cup 1 & \text{otherwise;} \end{cases}$$

A map A from S to the closed interval [0, 1] is called a fuzzy set in S. Let F(S) denote the set of all fuzzy sets in S. For any $A, B \in F(S)$, $A \subseteq B$ if and only if $A(x) \leqslant B(x)$, in the ordering of [0, 1] , for all $x \in S$.

Definition 2.1. For $A, B \in F(S)$. Then the product $A \circ B$ is defined as follows:

$$(A \circ B)(x) = \begin{cases} \sup\limits_{x=yz} \min\{A(y), B(z)\} & \text{for } y, z \in S,\ x = yz, \\ 0 & \text{for any } y, z \in S,\ x \neq yz; \end{cases}$$

for all $x \in S$. It is clear that $A \circ B \in F(S)$ (cf. [6]). And for

$C \in F(S)$, $A \circ (B \circ C) = (A \circ B) \circ C$. If $A \subseteq B$, then $A \circ C \subseteq B \circ C$ and $C \circ A \subseteq C \circ B$ (cf. [6, 7]). A and B are commutative with each other we mean $A \circ B = B \circ A$.

Pu and Liu gave the definition of a fuzzy point (cf. [8], Definition 2.1 and Definition 2.2). Clearly that every $A \in F(S)$,

$$A = \bigcup_{x_\lambda \in A} x_\lambda,$$

where $0 < \lambda \leqslant 1$, $x_\lambda \in A$ if and only if $x_\lambda \subseteq A$,

$$x_\lambda(y) = \begin{cases} \lambda & \text{if } y = x, \\ 0 & \text{if } y \neq x; \end{cases}$$

for all $y \in S$. For x_λ, $y_\mu \in F(S)$, $x_\lambda \subseteq y_\mu$ if and only if $x = y$ and $\lambda \leqslant \mu$.

Note. For all $x \in S$, $A_i \in F(S)$, $i \in I$ (indexing set),

$$(\bigcup_i A_i)(x) = \bigvee_i A_i(x) = \sup_i A_i(x), \quad (\bigcap_i A_i)(x) = \bigwedge_i A_i(x) = \inf_i A_i(x).$$

By [6], the following proposition is obvious.

Proposition 2.2. For any x_λ, $A \in F(S)$, then

$$x_\lambda \circ A = \bigcup_{y_\mu \in A} x_\lambda \circ y_\mu, \quad A \circ x_\lambda = \bigcup_{y_\mu \in A} y_\mu \circ x_\lambda.$$

Definition 2.3. $A \in F(S)$ is called a fuzzy subsemigroup of S if $A(xy) \geqslant \min\{A(x), A(y)\}$ for all $x, y \in S$, and is called a fuzzy left (right) ideal of S if $A(xy) \geqslant A(y)$ ($A(xy) \geqslant A(x)$) for all $x, y \in S$. $A \in F(S)$ is called a fuzzy ideal of S if it is both a fuzzy left and a fuzzy right ideal of S.

Proposition 2.4. $A \in F(S)$ is a fuzzy subsemigroup of S if and only if

$$A \circ A \subseteq A.$$

Proof. See Proposition 1.3 of [6].

In the following, let

$$I_R(S) = \{A \in F(S) \mid A \text{ is a fuzzy right ideal of } S\},$$
$$I_L(S) = \{A \in F(S) \mid A \text{ is a fuzzy left ideal of } S\}.$$

Proposition 2.5. Let A be a fuzzy subsemigroup of S, then for $a_\lambda \in A$,

$$\langle a_\lambda \rangle = \bigcap_{a_\lambda \in B \in I_L(S)\,(I_R(S))} B$$

is the smallest fuzzy left (right) ideal of S containing a_λ.

Proof. Directly from Definition 2.3.

3. Fuzzy left (right) ideals generated by fuzzy point

In the following, let δ_S be the characteristic function of S, then δ_S is the greatest fuzzy set in S and a fuzzy subsemigroup of S.

Definition 3.1. Let $a_\lambda \in \delta_S$, the smallest fuzzy left (right) ideal of S containing a_λ is called a fuzzy left (right) ideal of S generated by a_λ.

By Proposition 2.5, $<a_\lambda>$ is a fuzzy left (right) ideal of S generated by a_λ. We let $<a_\lambda>_L$ ($<a_\lambda>_R$) denote the fuzzy left (right) ideal of S generated by a_λ. The following two theorems give the construction of $<a_\lambda>_L$ ($<a_\lambda>_R$).

Theorem 3.2. Let $a_\lambda \in \delta_S$, then $<a_\lambda>_L = J$ ($<a_\lambda>_R = J$), where

$$J(x) = \begin{cases} \lambda & \text{if there exists } b \in S^1 \text{ such that } x = ba \quad (x = ab), \\ 0 & \text{otherwise;} \end{cases}$$

for all $x \in S$.

Proof. For any $x, y \in S$, if $J(y) = 0$, then $J(xy) \geqslant 0 = J(y)$; If $J(y) \neq 0$, then by the defining way of J, we have $J(y) = \lambda$ and there exists $b \in S^1$ such that $y = ba$. Since S^1 is a semigroup, $xb \in S^1$. Then $J(xy) = J(x(ba)) = J((xb)a) = \lambda \geqslant J(y)$, that is J is a fuzzy left ideal of S. Since $a = 1 \cdot a$, thus $J(a) = \lambda$, $a_\lambda \in J$. Let I be a fuzzy left ideal of S containing a_λ, then $I(a) \geqslant \lambda$. For all $x \in S$, if $J(x) = 0$, then $I(x) \geqslant 0 = J(x)$; Otherwise $J(x) = \lambda$ and there exists $b \in S^1$ such that $x = ba$. If $b = 1$, then $I(x) = I(ba) = I(a) \geqslant \lambda = J(x)$; If $b \neq 1$, then $b \in S$, $I(x) = I(ba) \geqslant I(a) \geqslant \lambda = J(x)$. So $J \subseteq I$. Hence J is the smallest fuzzy left ideal of S containing a_λ. By Definition 3.1, $<a_\lambda>_L = J$.

In the same manner we can prove the right case.

Theorem 3.3. Let $a_\lambda \in \delta_S$, then

$$<a_\lambda>_L = a_\lambda \cup \delta_S \circ a_\lambda \quad (<a_\lambda>_R = a_\lambda \cup a_\lambda \circ \delta_S).$$

Proof. By Proposition 2.2,

$$\delta_S \circ a_\lambda = \bigcup_{x_\mu \in \delta_S} x_\mu \circ a_\lambda.$$

For any $x_\mu \circ a_\lambda \in \delta_S \circ a_\lambda$, $<a_\lambda>_L(xa) \geqslant <a_\lambda>_L(a)$ since $<a_\lambda>_L$ is a fuzzy left ideal of S. But $<a_\lambda>_L(a) \geqslant \lambda$, so $<a_\lambda>_L(xa) \geqslant \lambda$, thus $(xa)_\lambda \in <a_\lambda>_L$. Since

$$x_\mu \circ a_\lambda = (xa)_{\inf\{\lambda, \mu\}} \subseteq (xa)_\lambda \in <a_\lambda>_L,$$

that is $x_\mu \circ a_\lambda \in <a_\lambda>_L$. So

$$\delta_S \circ a_\lambda = \bigcup_{x_\mu \in \delta_S} x_\mu \circ a_\lambda \subseteq <a_\lambda>_L,$$

that is $a_\lambda \cup \delta_S \circ a_\lambda \subseteq <a_\lambda>_L$. On the other hand, let

$$<a_\lambda>_L = \bigcup_{x_\mu \in <a_\lambda>_L} x_\mu,$$

then for any $x_\mu \in <a_\lambda>_L$, we have $<a_\lambda>_L(x) \geqslant \mu > 0$. So by Theorem 3.2, $<a_\lambda>_L(x) = \lambda$ and there exists $b \in S^1$ such that $x = ba$. If $b = 1$, then $x = a$ and $x_\mu = a_\mu \subseteq a_\lambda \in a_\lambda \cup \delta_S \circ a_\lambda$ since $\lambda \geqslant \mu$. If $b \neq 1$, then $b \in S$

and $a_\mu \subseteq a_\lambda$. So $\delta_S \circ a_\mu \subseteq \delta_S \circ a_\lambda$. Then $b_\mu \in \delta_S$ and

$$x_\mu = b_\mu \circ a_\mu \in \delta_S \circ a_\mu \subseteq \delta_S \circ a_\lambda \subseteq a_\lambda \cup \delta_S \circ a_\lambda,$$

that is $\langle a_\lambda \rangle_L \subseteq a_\lambda \cup \delta_S \circ a_\lambda$. This follows that

$$\langle a_\lambda \rangle_L = a_\lambda \cup \delta_S \circ a_\lambda.$$

Similarly we can prove $\langle a_\lambda \rangle_R = a_\lambda \cup a_\lambda \circ \delta_S$.

Definition 3.4. Let $a_\lambda \in \delta_S$, the smallest fuzzy ideal of S containing a_λ is called a fuzzy ideal of S generated by a_λ. Denoted by $\langle a_\lambda \rangle$.

Theorem 3.5. Let $a_\lambda \in \delta_S$, then $\langle a_\lambda \rangle = J$, where

$$J(x) = \begin{cases} \lambda & \text{if there exist } b, c \in S^1 \text{ such that } x = bac, \\ 0 & \text{otherwise;} \end{cases}$$

for all $x \in S$.

Theorem 3.6. Let $a_\lambda \in \delta_S$, then

$$\langle a_\lambda \rangle = a_\lambda \cup a_\lambda \circ \delta_S \cup \delta_S \circ a_\lambda \cup \delta_S \circ a_\lambda \circ \delta_S .$$

The proofs of Theorem 3.5, 3.6, similar to Theorem 3.2, 3.3 respectively, are omitted.

4. Fuzzy regular elements and fuzzy inverse subsemigroups

Proposition 4.1. Let A be a fuzzy subsemigroup of S, then for any $a_\lambda \in A$, $a_\lambda \circ A \subseteq A$, $A \circ a_\lambda \subseteq A$.

Proof. Using Proposition 2.2 and 2.4.

Proposition 4.2. Let A be a fuzzy subsemigroup of S, then for any $a_\lambda \in A$, $A \circ a_\lambda \subseteq \langle a_\lambda \rangle_L$ $(a_\lambda \circ A \subseteq \langle a_\lambda \rangle_R)$.

Proof. By $A \subseteq \delta_S$ and Theorem 3.3.

Definition 4.3. Let $A \in F(S)$, for $x_\lambda \in A$, x_λ is called a fuzzy regular element of A, or simply regular, if and only if there exists $y_\lambda \in A$ such that $x_\lambda = x_\lambda \circ y_\lambda \circ x_\lambda$.

It is convenient to set $A \circ A = A^2$ in the following.

Proposition 4.4. Let A be a fuzzy subsemigroup of S and a_λ a fuzzy regular element of A, then

(1) $a_\lambda \subseteq a_\lambda \circ A \subseteq a_\lambda \circ A^2 \subseteq \cdots \subseteq a_\lambda \circ A^n \subseteq \cdots \subseteq A$;

(2) $a_\lambda \circ A = a_\lambda \circ A^2$.

Proof. Since a_λ is a fuzzy regular element of A, there exists $y_\lambda \in A$ such that $a_\lambda \circ y_\lambda \circ a_\lambda = a_\lambda$, then $a = aya$,

$$(a_\lambda \circ A)(a) = \sup_{a=uv} \min\{a_\lambda(u), \ A(v)\} \geqslant \min\{a_\lambda(a), \ A(ya)\}.$$

Since A is a fuzzy subsemigroup of S, we have $A(ya) \geqslant \min\{A(a), A(y)\} \geqslant \min\{\lambda, \lambda\} = \lambda$. Then $(a_\lambda \circ A)(a) \geqslant \lambda$, that is $a_\lambda \subseteq a_\lambda \circ A$. Hence

$$a_\lambda \subseteq a_\lambda \circ A \subseteq a_\lambda \circ A^2 \subseteq \cdots \subseteq a_\lambda \circ A^n \subseteq \cdots .$$

By Proposition 2.4 and 4.1,

$$a_\lambda \subseteq a_\lambda \circ A \subseteq \cdots \subseteq a_\lambda \circ A^n \subseteq \cdots \subseteq A.$$

Again using Proposition 2.4, $A^2 \subseteq A$, so $a_\lambda \circ A = a_\lambda \circ A^2$.

Proposition 4.5. Let A be a fuzzy subsemigroup of S and a_λ a fuzzy regular element of A. Then

(1) $a_\lambda \subseteq A \circ a_\lambda \subseteq A^2 \circ a_\lambda \subseteq \cdots \subseteq A^n \circ a_\lambda \subseteq \cdots \subseteq A$;

(2) $A \circ a_\lambda = A^2 \circ a_\lambda$.

Proof. Similar to Proposition 4.4.

Remark. The example below shows that generally:

(1) In Theorem 3.3, a fuzzy set A $(\neq \delta_S)$ in S can not be instead of δ_S even if a_λ is a fuzzy regular element.

(2) For $a_\lambda \in A$ $(\neq \delta_S)$, $\langle a_\lambda \rangle_L \not\subseteq A$ $(\langle a_\lambda \rangle_R \not\subseteq A)$.

Example. Let S = {x, y}, the binary operation on S is defined as follows:

•	x	y
x	x	y
y	y	y

Evidently S is a semigroup with respect to the binary operation as above. Let $A = x_{0.7}$, then $A \in F(S)$. Since

$$A(xx) = A(x) \geqslant \min\{A(x), A(x)\},$$
$$A(xy) = A(y) \geqslant \min\{A(x), A(y)\},$$
$$A(yx) = A(xy),$$
$$A(yy) = A(y) \geqslant \min\{A(y), A(y)\}.$$

So A is a fuzzy subsemigroup of S. Obviously $a_\lambda = x_{0.5} \in A$ and a_λ is a fuzzy regular element of A since $x_{0.5} \circ x_{0.5} \circ x_{0.5} = (xxx)_{0.5} = x_{0.5}$. By Proposition 4.5,

$$a_\lambda \cup A \circ a_\lambda = A \circ a_\lambda = x_{0.7} \circ x_{0.5} = x_{0.5},$$

then $(a_\lambda \cup A \circ a_\lambda)(y) = 0$. Since $x = xx$, $y = yx$, then by Theorem 3.2,

$$\langle a_\lambda \rangle_L(x) = 0.5, \quad \langle a_\lambda \rangle_L(y) = 0.5.$$

Hence $a_\lambda \cup A \circ a_\lambda \neq \langle a_\lambda \rangle_L$. Let $B \in F(S)$ hold $a_\lambda \subseteq B \subseteq A$, then B can not be a fuzzy left ideal of S for ever, this follows $\langle a_\lambda \rangle_L \not\subseteq A$. Indeed, for $x, y \in S$,

$$B(yx) = B(y) \leqslant A(y) = 0, \text{ but } B(x) \geqslant a_\lambda(x) = 0.5.$$

That is $B(yx) < B(x)$, B is not a fuzzy left ideal of S.

Definition 4.6. Let $A \in F(S)$, for $e_\lambda \in A$, e_λ is called a fuzzy idempotent element of A if and only if $e_\lambda \circ e_\lambda = e_\lambda$.

Proposition 4.7. Let A be a fuzzy subsemigroup of S and a_λ is a fuzzy regular element of A. Then there exist fuzzy idempotent elements e_λ, $f_\mu \in A$

such that:

(1) $a_\lambda \cup A \circ a_\lambda = e_\lambda \cup A \circ e_\lambda$;

(2) $a_\lambda \cup a_\lambda \circ A = f_\mu \cup f_\mu \circ A$.

Proof. Obviously a fuzzy idempotent element is also a fuzzy regular element. So by Proposition 4.5, we only need to prove there exists a fuzzy idempotent element $e_\lambda \in A$ such that $A \circ a_\lambda = A \circ e_\lambda$. Firstly, since a_λ is a fuzzy regular element of A, there exists $x_\lambda \in A$ such that $a_\lambda \circ x_\lambda \circ a_\lambda = a_\lambda$. Let $e_\lambda = x_\lambda \circ a_\lambda$, then

$$e_\lambda \circ e_\lambda = (x_\lambda \circ a_\lambda) \circ (x_\lambda \circ a_\lambda) = x_\lambda \circ (a_\lambda \circ x_\lambda \circ a_\lambda) = e_\lambda,$$

that is e_λ is a fuzzy idempotent element of A. And $a_\lambda \circ e_\lambda = a_\lambda \circ x_\lambda \circ a_\lambda = a_\lambda$. By Proposition 2.2, let

$$A \circ a_\lambda = \bigcup_{y_\mu \in A} y_\mu \circ a_\lambda, \quad A \circ e_\lambda = \bigcup_{y_\mu \in A} y_\mu \circ e_\lambda.$$

For any $y_\mu \circ a_\lambda \in A \circ a_\lambda$, $y_\mu \circ a_\lambda = y_\mu \circ (a_\lambda \circ e_\lambda) = (y_\mu \circ a_\lambda) \circ e_\lambda$, thus $y_\mu \circ a_\lambda \in A \circ e_\lambda$ since $y_\mu \circ a_\lambda \in A$, so $A \circ a_\lambda \subseteq A \circ e_\lambda$. On the other hand, for any $y_\mu \circ e_\lambda \in A \circ e_\lambda$, $y_\mu \circ e_\lambda = y_\mu \circ (x_\lambda \circ a_\lambda) = (y_\mu \circ x_\lambda) \circ a_\lambda$, thus $y_\mu \circ e_\lambda \in A \circ a_\lambda$ since $y_\mu \circ x_\lambda \in A$, so $A \circ e_\lambda \subseteq A \circ a_\lambda$. Consequently, $A \circ a_\lambda = A \circ e_\lambda$. Similarly we can prove (2).

Proposition 4.8. Let A be a fuzzy subsemigroup of S and $a_\lambda \in A$, where S is a finite semigroup. Then a_λ is a fuzzy regular element of A if and only if there exists a fuzzy idempotent element $e_\lambda \in A$ such that $a_\lambda \cup A \circ a_\lambda = e_\lambda \cup A \circ e_\lambda$ ($a_\lambda \cup a_\lambda \circ A = e_\lambda \cup e_\lambda \circ A$).

Proof. By Proposition 4.7 we only need to prove the sufficiency. Suppose there exists a fuzzy idempotent $e_\lambda \in A$ such that $a_\lambda \cup A \circ a_\lambda = e_\lambda \cup A \circ e_\lambda$. Since $a_\lambda \in a_\lambda \cup A \circ a_\lambda = e_\lambda \cup A \circ e_\lambda$, $a_\lambda \neq e_\lambda$, then $a_\lambda \in A \circ e_\lambda$, that is $(A \circ e_\lambda)(a) \geqslant \lambda$. Thus

$$(A \circ e_\lambda)(a) = \sup_{a=xy} \min\{A(x), e_\lambda(y)\} = \sup_{a=xe} \min\{A(x), \lambda\}.$$

Since S is finite, we have $(A \circ e_\lambda)(a) = \max_{a=xe} \min\{A(x), \lambda\}$.

Then there exists $x \in S$ such that $a = xe$ and $\min\{A(x), \lambda\} \geqslant \lambda$, thus $A(x) \geqslant \lambda$, that is $x_\lambda \in A$. Similarly, by $e_\lambda \in a_\lambda \cup A \circ a_\lambda$, $a_\lambda \neq e_\lambda$, we can prove there exists $y \in S$ such that $e = ya$ and $y_\lambda \in A$. This follows that

$$a_\lambda \circ y_\lambda \circ a_\lambda = a_\lambda \circ e_\lambda = (x_\lambda \circ e_\lambda) \circ e_\lambda = x_\lambda \circ (e_\lambda \circ e_\lambda) = x_\lambda \circ e_\lambda = a_\lambda.$$

That is a_λ is a fuzzy regular element of A.

Similarly we can verify the other case.

Theorem 4.9. For $a_\lambda \in \delta_S$, a_λ is a fuzzy regular element of δ_S if and only if there exists a fuzzy idempotent element $e_\lambda \in \delta_S$ such that

$$a_\lambda \cup \delta_S \circ a_\lambda = e_\lambda \cup \delta_S \circ e_\lambda \quad (a_\lambda \cup a_\lambda \circ \delta_S = e_\lambda \cup e_\lambda \circ \delta_S).$$

Proof. By Proposition 4.7, we only need to prove the sufficiency. Assume there exists a fuzzy idempotent element e_λ of δ_S such that $a_\lambda \cup \delta_S \circ a_\lambda = e_\lambda \cup \delta_S \circ e_\lambda$. From Theorem 3.3 we have $\langle a_\lambda \rangle_L = \langle e_\lambda \rangle_L$. Since $a_\lambda \in \langle a_\lambda \rangle_L = \langle e_\lambda \rangle_L$, then $\langle e_\lambda \rangle_L(a) \geqslant \lambda$. From Theorem 3.2, $\langle e_\lambda \rangle_L(a) = \lambda$ and there exists $x \in S^1$ such that $a = xe$. Similarly from $e_\lambda \in \langle e_\lambda \rangle_L = \langle a_\lambda \rangle_L$, we can prove there exists $y \in S^1$ such that $e = ya$. If $x = 1$ or $y = 1$, then $a = e$ and $a_\lambda = e_\lambda$ is a fuzzy regular element of δ_S; If $x \neq 1$ and $y \neq 1$, then $x, y \in S$, $a_\lambda \circ y_\lambda \circ a_\lambda = a_\lambda \circ e_\lambda = (x_\lambda \circ e_\lambda) \circ e_\lambda = x_\lambda \circ (e_\lambda \circ e_\lambda) = x_\lambda \circ e_\lambda = a_\lambda$ while $y_\lambda \in \delta_S$. Hence a_λ is a fuzzy regular element of δ_S.

Definition 4.10. Let A be a fuzzy subsemigroup of S. For x_λ, $y_\mu \in A$, x_λ and y_μ are called inverses of each other if $x_\lambda = x_\lambda \circ y_\mu \circ x_\lambda$ and $y_\mu = y_\mu \circ x_\lambda \circ y_\mu$. We also call y_μ an inverse of x_λ and x_λ an inverse of y_μ.

By Definition 4.10, it is easy to see that $\lambda = \mu$ and the following proposition is obvious.

Proposition 4.11. Let A be a fuzzy subsemigroup of S. If a_λ is a fuzzy regular element of A, then a_λ has at least one inverse in A.

Proposition 4.12. If e_λ, f_μ, $e_\lambda \circ f_\mu$, $f_\mu \circ e_\lambda$ are all fuzzy idempotent elements of a fuzzy subsemigroup of S, then $e_\lambda \circ f_\mu$ and $f_\mu \circ e_\lambda$ are inverses of each other.

The proof is immediate.

Definition 4.13. A fuzzy subsemigroup A of S is called a fuzzy regular subsemigroup if every element of A is regular.

It is easy to verify that Definition 4.13 is equivalent to Definition 2.1 of [5].

Definition 4.14. A fuzzy subsemigroup A of S is called a fuzzy inverse subsemigroup if and only if every element of A has a unique inverse in A.

Proposition 4.15. If A is a fuzzy regular subsemigroup of S, and any two fuzzy idempotent elements of A commute with each other, then for any $a_\lambda \in A$, there exist fuzzy idempotent elements $e_\lambda \in A$ and $f_\mu \in A$ such that $a_\lambda \cup A \circ a_\lambda = e_\lambda \cup A \circ e_\lambda$ and $a_\lambda \cup a_\lambda \circ A = f_\mu \cup f_\mu \circ A$, respectively. And e_λ and f_μ are both unique.

Proof. By Proposition 4.7, there exists a fuzzy idempotent element $e_\lambda \in A$ such that $a_\lambda \cup A \circ a_\lambda = e_\lambda \cup A \circ e_\lambda$. Assume there exists another fuzzy idempotent element $e^*_\lambda \in A$ holds $a_\lambda \cup A \circ a_\lambda = e^*_\lambda \cup A \circ e^*_\lambda$. Then $e^*_\lambda \cup A \circ e^*_\lambda = e_\lambda \cup A \circ e_\lambda$. By Proposition 4.5, $e^*_\lambda \in A \circ e^*_\lambda$, $e_\lambda \in A \circ e_\lambda$, then $A \circ e^*_\lambda = A \circ e_\lambda$. From $e^*_\lambda \in A \circ e_\lambda$ we have $(A \circ e_\lambda)(e^*) \geqslant \lambda$, that is

$$(A \circ e_\lambda)(e^*) = \sup_{e^* = xy} \min\{A(x),\ e_\lambda(y)\} \geqslant \lambda.$$

Thus there exist $x, y \in S$ such that $e^* = xy$, $A(x) \neq 0$ and $e_\lambda(y) = \lambda$. Then $y = e$, hence $e^* = xe$, $e^*_\lambda = x_\lambda \circ e_\lambda = x_\lambda \circ (e_\lambda \circ e_\lambda) = (x_\lambda \circ e_\lambda) \circ e_\lambda = e^*_\lambda \circ e_\lambda$. Similarly from $e_\lambda \in A \circ e^*_\lambda$, we can prove $e_\lambda \circ e^*_\lambda = e_\lambda$. By hypothesis

$e_\lambda \circ e^*_\lambda = e^*_\lambda \circ e_\lambda$, and hence $e_\lambda = e^*_\lambda$. Similarly we can prove the fuzzy idempotent element f_μ of A satisfying $a_\lambda \cup a_\lambda \circ A = f_\mu \cup f_\mu \circ A$ is unique.

Proposition 4.16. If A is a fuzzy inverse subsemigroup of S, then A is a fuzzy regular subsemigroup, and any two fuzzy idempotent elements of A commute with each other.

Proof. Evidently A is a fuzzy regular subsemigroup of S. Let e_λ and f_μ be any two fuzzy idempotent elements of A. Since $e_\lambda \circ f_\mu \in A$, let a_v be the unique inverse of $e_\lambda \circ f_\mu$. Then

$$(e_\lambda \circ f_\mu) \circ a_v \circ (e_\lambda \circ f_\mu) = e_\lambda \circ f_\mu, \quad a_v \circ (e_\lambda \circ f_\mu) \circ a_v = a_v.$$

Let $b_u = a_v \circ e_\lambda$, then

$$\begin{aligned}(e_\lambda \circ f_\mu) \circ b_u \circ (e_\lambda \circ f_\mu) &= e_\lambda \circ f_\mu \circ a_v \circ e_\lambda \circ e_\lambda \circ f_\mu \\ &= e_\lambda \circ f_\mu \circ a_v \circ e_\lambda \circ f_\mu \\ &= e_\lambda \circ f_\mu, \\ b_u \circ (e_\lambda \circ f_\mu) \circ b_u &= a_v \circ e_\lambda \circ (e_\lambda \circ f_\mu) \circ a_v \circ e_\lambda \\ &= a_v \circ e_\lambda \circ f_\mu \circ a_v \circ e_\lambda \\ &= a_v \circ e_\lambda \\ &= b_u.\end{aligned}$$

Hence b_u is also an inverse of $e_\lambda \circ f_\mu$. By the uniqueness of a_v, we have $a_v \circ e_\lambda = b_u = a_v$. Similarly we can show $f_\mu \circ a_v = a_v$. Hence $a_v \circ a_v = (a_v \circ e_\lambda) \circ (f_\mu \circ a_v) = a_v \circ (e_\lambda \circ f_\mu) \circ a_v = a_v$.

That is a_v is a fuzzy idempotent element of A. But a fuzzy idempotent element of A is an inverse of itself, and, again using the uniqueness of a_v, we conclude that $a_v = e_\lambda \circ f_\mu$. Hence $e_\lambda \circ f_\mu$ is a fuzzy idempotent element of A. By the foregoing, $f_\mu \circ e_\lambda$ are also a fuzzy idempotent of A. By Proposition 4.12, $e_\lambda \circ f_\mu$ and $f_\mu \circ e_\lambda$ are inverses of each other. Thus $e_\lambda \circ f_\mu$ and $f_\mu \circ e_\lambda$ are both inverses of $e_\lambda \circ f_\mu$, and so $e_\lambda \circ f_\mu = f_\mu \circ e_\lambda$.

Theorem 4.17. The following three conditions are equivalent:

(i) δ_S is a fuzzy regular subsemigroup, and any two fuzzy idempotent elements of δ_S commute with each other;

(ii) For any $a_\lambda \in \delta_S$, there exist fuzzy idempotents $e_\lambda, f_\mu \in \delta_S$ such that $\langle a_\lambda \rangle_L = \langle e_\lambda \rangle_L$, $\langle a_\lambda \rangle_R = \langle f_\mu \rangle_R$. And e_λ and f_μ are both unique;

(iii) δ_S is a fuzzy inverse subsemigroup.

Proof. By Proposition 4.15 and 4.16, we only need to show that (ii) implies (iii). By Theorem 4.9, every $a_\lambda \in \delta_S$ is a fuzzy regular element of δ_S. Therefore in the following, we prove every $a_\lambda \in \delta_S$ has a unique fuzzy inverse in δ_S. Let x_λ, y_λ are both inverses of a_λ. Then

$$a_\lambda \circ x_\lambda \circ a_\lambda = a_\lambda, \quad x_\lambda \circ a_\lambda \circ x_\lambda = x_\lambda,$$
$$a_\lambda \circ y_\lambda \circ a_\lambda = a_\lambda, \quad y_\lambda \circ a_\lambda \circ y_\lambda = y_\lambda.$$

By Proposition 4.1, For any $b_\lambda \in \delta_S$, $b_\lambda \circ \delta_S \subseteq \delta_S$. Hence

$$a_\lambda \circ x_\lambda \circ \delta_S = a_\lambda \circ (x_\lambda \circ \delta_S) \subseteq a_\lambda \circ \delta_S,$$

$$a_\lambda \circ \delta_s = (a_\lambda \circ x_\lambda \circ a_\lambda) \circ \delta_s = (a_\lambda \circ x_\lambda) \circ (a_\lambda \circ \delta_s) \subseteq a_\lambda \circ x_\lambda \circ \delta_s.$$

Therefore $a_\lambda \circ x_\lambda \circ \delta_s = a_\lambda \circ \delta_s$. Similarly we have $a_\lambda \circ y_\lambda \circ \delta_s = a_\lambda \circ \delta_s$. Then $a_\lambda \circ x_\lambda \circ \delta_s = a_\lambda \circ y_\lambda \circ \delta_s$. Hence $a_\lambda \circ x_\lambda \cup a_\lambda \circ x_\lambda \circ \delta_s = a_\lambda \circ y_\lambda \cup a_\lambda \circ y_\lambda \circ \delta_s$, that is $\langle a_\lambda \circ x_\lambda \rangle_L = \langle a_\lambda \circ y_\lambda \rangle_L$. By (ii), $a_\lambda \circ x_\lambda = a_\lambda \circ y_\lambda$ since $a_\lambda \circ x_\lambda$ and $a_\lambda \circ y_\lambda$ are both fuzzy idempotent elements of δ_s. Similarly we can show $x_\lambda \circ a_\lambda = y_\lambda \circ a_\lambda$. Hence $x_\lambda = x_\lambda \circ a_\lambda \circ x_\lambda = (y_\lambda \circ a_\lambda) \circ x_\lambda = y_\lambda \circ a_\lambda \circ y_\lambda = y_\lambda$. This follows that δ_s is a fuzzy inverse subsemigroup of S.

Proposition 4.18. If S is a finite semigroup, then the following three conditions are equivalent:

(i) A is a fuzzy regular subsemigroup of S, and any two fuzzy idempotent elements of A commute with each other;

(ii) For any $a_\lambda \in A$, there exist fuzzy idempotent elements $e_\lambda \in A$ and $f_\mu \in A$ such that $a_\lambda \cup A \circ a_\lambda = e_\lambda \cup A \circ e_\lambda$ and $a_\lambda \cup a_\lambda \circ A = f_\mu \cup f_\mu \circ A$, respectively. And e_λ and f_μ are both unique;

(iii) A is a fuzzy inverse subsemigroup of S.

Proof. Using Proposition 4.8 and the proof way similar to Theorem4.17, we can prove Proposition 4.18.

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